DIGITAL IMAGE PROCESSING
15CS753

MODULE-3

Image Enhancement In The Frequency Domain

- Module – 3 8 Hours
- Image Enhancement In Frequency Domain:
- Introduction,
- Fourier Transform,
- Discrete Fourier Transform (DFT),
- properties of DFT,
- Discrete Cosine Transform (DCT),
- Image filtering in frequency domain..

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- Introduction
- These are proposed by French mathematician Joseph Fourier
- His contribution basically states that, any periodic function can be expressed as sum of sines and/or cosines of different frequencies each multiplied by a different coefficient
- This sum is termed as Fourier series.
- Irrespective of how complicated the function is, if it is periodic and if it satisfies some mathematical conditions, it can be represented by such a sum
- The functions which are not periodic but whose area under the curve is finite can be expressed as integral of sines and/or cosines multiplied by the weighting function.
- This is called as Fourier Transform and is more widely used than Fourier series

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- The important characteristic of both F.S. and F.T. is that a function expressed either in F.S. or F.T. can be reconstructed (recovered) via a reverse process with no loss of information
- This allows us to work in Fourier domain and then return to the original domain without loss of information
- Since we are dealing with images which are functions of finite duration, we will be using Fourier Transform as a tool
- Basic concepts:
- A complex number is defined as C= R+jI
- Where R and I are real numbers and i is imaginary number equal to square root of -1.
- .

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- Polar representation of complex numbers

- .

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- Fourier Series
- We know that, a function f(t) of a continuous variable t, periodic with period T can be expresses as sum of sines and cosines multiplied by appropriate coefficients
- This sum known as Fourier series can be formulated as
- .
- .

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t} \quad \text{....... (1)}$$

- Where the coefficients are
- $$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-j\frac{2\pi n}{T}t}\,dt \qquad \text{for } n = 0, \pm 1, \pm 2, \ldots \quad (2)$$

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- Impulses and their Sifting:
- Before studying the Fourier Transforms we need to learn about impulses and their sifting property
- A unit impulse of a continuous variable t located at t=0 denoted δ(t) is defined by
- $$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}$$ ….3-a.
- This is constrained to satisfy the identity
- $$\int_{-\infty}^{\infty} \delta(t)dt = 1$$ ….3.b
- This means that, at time t, impulse can be viewed as a spike of infinite amplitude and zero duration having unit area

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- An impulse has the sifting property w.r.t. integration
- $$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0) \qquad \ldots(4)$$
- .provided that f(t) is continuous at t=0
- Sifting yields the value of the function f(t), at the location of the impulse. (previous equation, at t=0)
- A more general statement of the sifting property involves the impulse located at an arbitrary point t0, denoted by δ(t-t0).
- Now sifting property results in

$$\int_{-infty}^{\infty} f(t)\delta(t-t_0)dt = f(t_0) \qquad \ldots..(5)$$

- This yields the value of the function at the location t0.

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- Suppose if f(t)= cos(t), using the impulse δ(t-π) in the equation (5) we get the result as f(π)=cos(π) = -1
- Let x represent a discrete variable.
- The unit discrete impulse δ(x) serves the same purpose in the context of discrete systems, as the impulse δ(t) does while working with continuous variables
- δ(x) is defined by
- .

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{.....(6)}$$

- This also satisfies the discrete equivalent of the equation (3.b)

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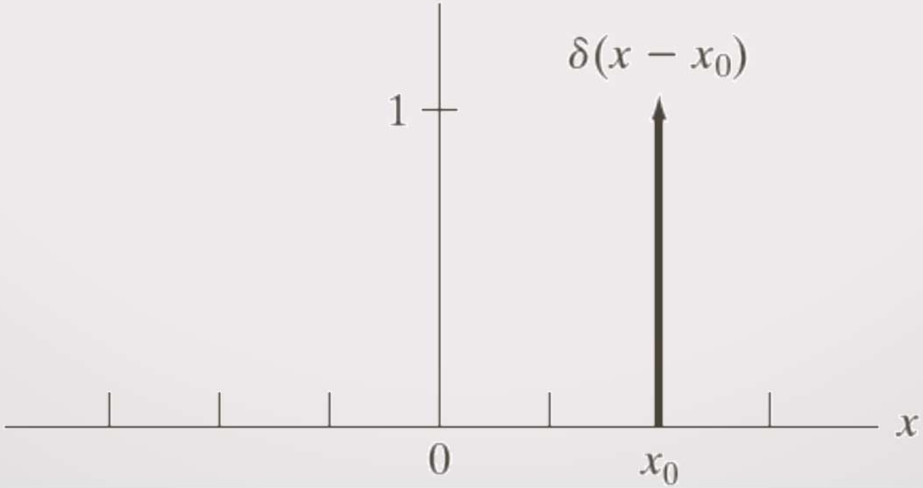
- Sifting property of the discrete impulse is given by
- $$\sum_{x=-\infty}^{\infty} f(x)\delta(x) = f(0) \quad \ldots(7)$$
- More general form of sifting property can be written as
- .
  $$\sum_{x=-\infty}^{\infty} f(x)\delta(x-x_0) = f(x_0)$$
- ….(8)
- Here also it is clear that, sifting gives the value of the function f(x) at the location of impulse
- Schematically unit discrete impulse can be shown as below



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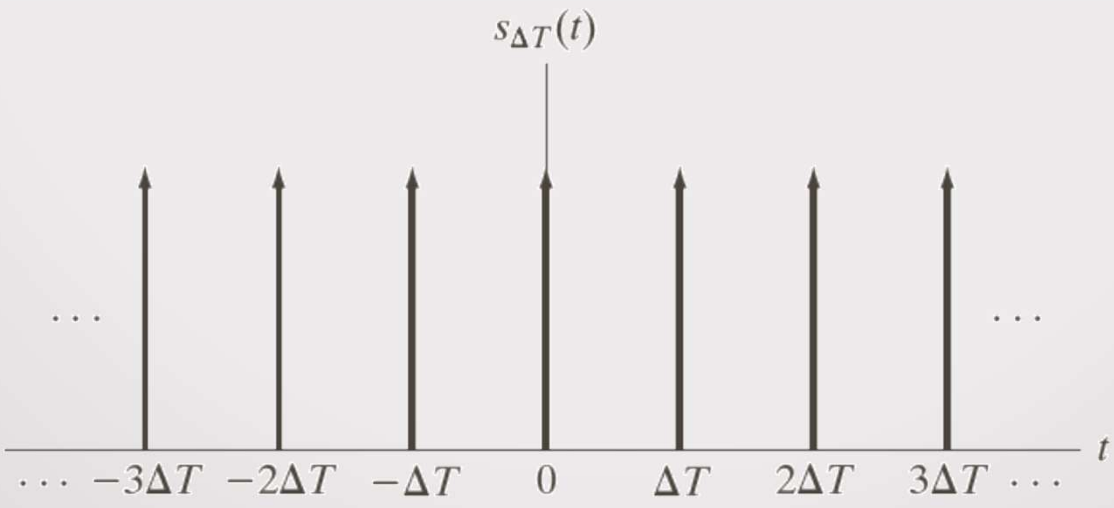
- Impulse train:
- This is defined as the sum of infinitely many periodic impulses separated by ΔT units
- Mathematically we can write this as
- $$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T) \quad \text{....(9)}$$
- Schematically we have impulse train as shown below
- .



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- Fourier transforms of a function with one variable
- Given single variable continuous function f(t) of a continuous variable t
- Fourier transform F(u) is given by
- $$\mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt$$ ….(10)
- Where μ is also a continuous variable
- Though we see two variables t and μ, since t gets integrated out, we can see that F.T. is a function of only one variable
- For simplicity let us denote F.T. as Ƒ{f(t)} = F(μ)
- Thus Fourier transform of f(t) is given by
- $$F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt$$ …..(11)

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- Suppose if we are given with F(μ) we can get back f(t) by using inverse Fourier Transform
- i.e.
- $$f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t} d\mu \quad \ldots (12)$$
- Equations 11 and 12 are called as Fourier transform pairs
- Using Euler's formula we can write equation 11 as .

$$F(\mu) = \int_{-\infty}^{\infty} f(t)[\cos(2\pi\mu t) - j\sin(2\pi\mu t)]dt$$

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- If f(t) is real, its transform is complex.
- Note that the Fourier transform is an expansion of f(t) multiplied by sinusoidal terms whose frequencies are determined by the values of μ (variable t is integrated out).
- Because the only variable left after integration is frequency, we say that the domain of the Fourier transform is the frequency domain.
- In our discussion, t can represent any continuous variable, and the units of the frequency variable μ depend on the units of t.
- For example, if t represents time in seconds, the units of μ are cycles/sec or Hertz (Hz).
- If t represents distance in meters, then the units of μ are cycles/meter, and so on.
- In other words, the units of the frequency domain are cycles per unit of the independent variable of the input function..

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- Fourier Series
- We know that, a function f(t) of a continuous variable t, periodic with period T can be expresses as sum of sines and cosines multiplied by appropriate coefficients
- This sum known as Fourier series can be formulated as
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$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t}$$

- . ……. (1)
- Where the coefficients are
- $$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-j\frac{2\pi n}{T}t}\,dt \qquad \text{for } n = 0, \pm 1, \pm 2, \ldots$$ (2)

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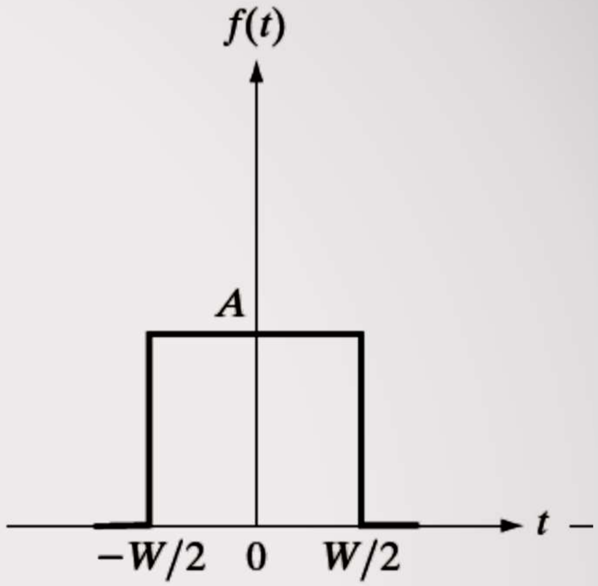
- Computing Fourier transform
- Consider the function shown below
- Using the equation (11) we can write the



$$F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t}\, dt$$

$$= \int_{-W/2}^{W/2} A e^{-j2\pi\mu t}\, dt$$

- Applying the integration rules we get .

$$= \frac{-A}{j2\pi\mu} \left[e^{-j2\pi\mu t} \right]_{-W/2}^{W/2}$$

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- Further simplifying we get

$$= \frac{-A}{j2\pi\mu}\left[e^{-j\pi\mu W} - e^{j\pi\mu W}\right]$$

$$= \frac{A}{j2\pi\mu}\left[e^{j\pi\mu W} - e^{-j\pi\mu W}\right]$$

- Using the identity sin θ = (ejθ – e-jθ)/2j we can further simplify this as
- F(u) = $AW \frac{\sin(\pi\mu W)}{(\pi\mu W)}$

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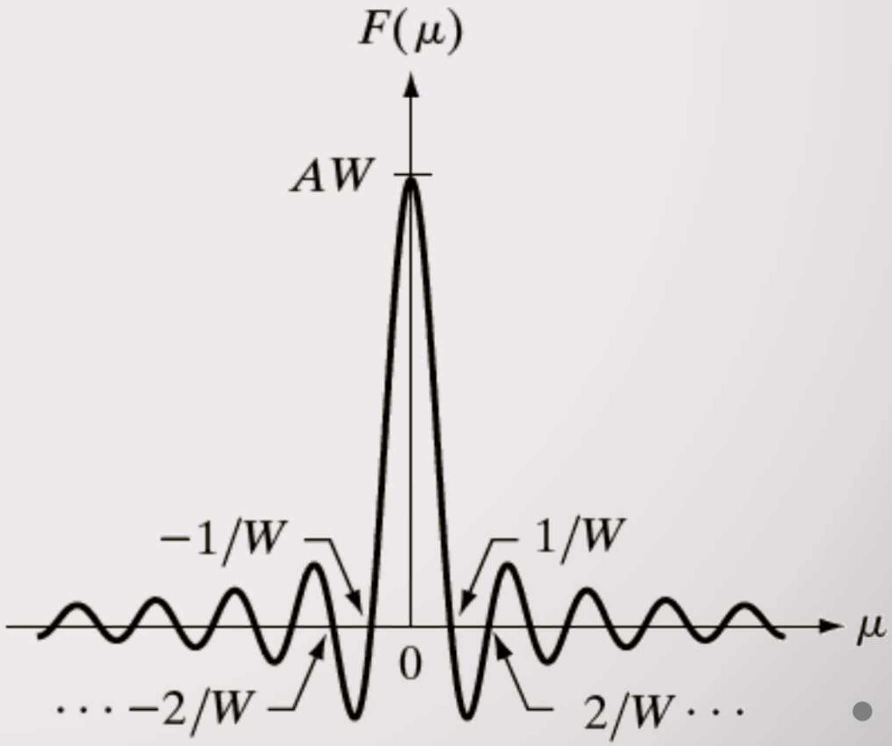
- In this equation we can see that, the complex terms of Fourier transform are nicely combined into a sine function
- The result in the last step of the previous expression is known as sinc function

$$\text{sinc}(m) = \frac{\sin(\pi m)}{(\pi m)}$$

- where sinc(0) = 1, and sinc(m) = 0 for all other integer values of m.
- The plot of F(μ) is shown below



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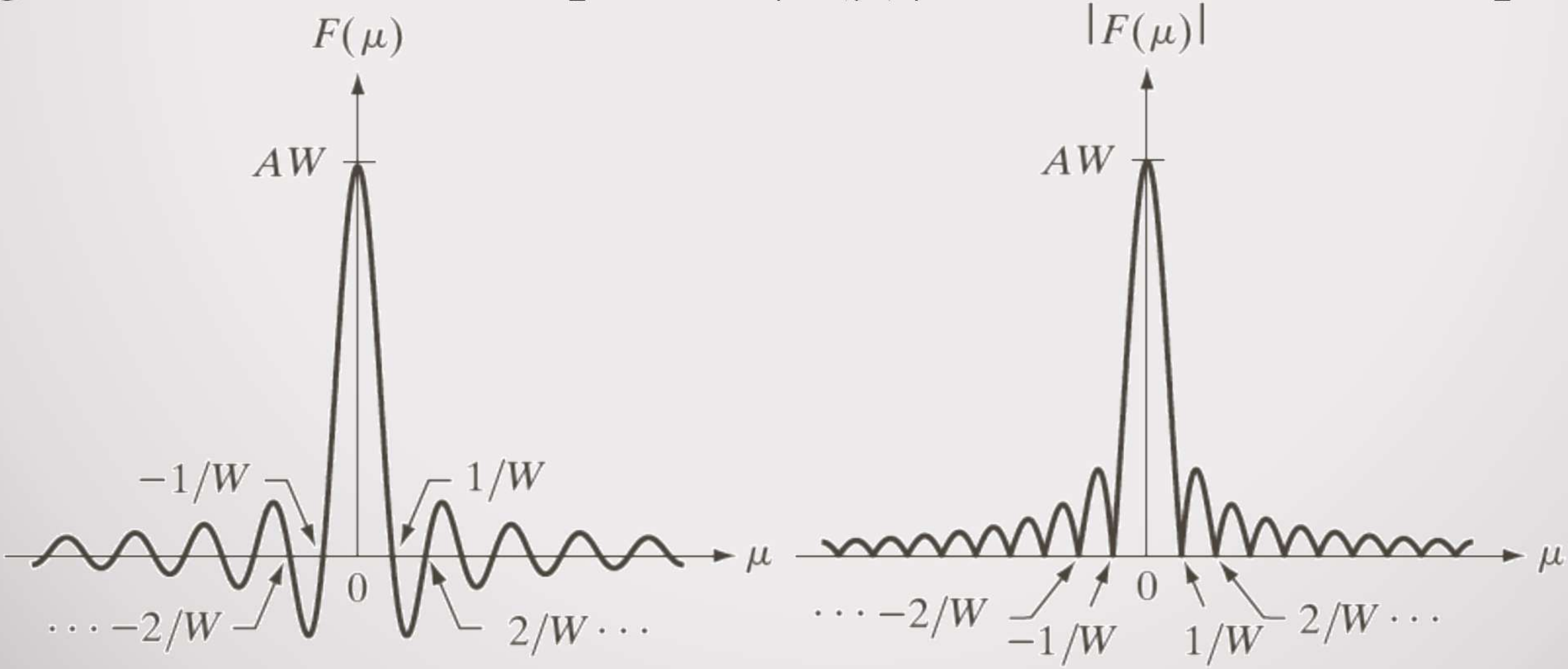
- In general, the Fourier transform contains complex terms, and it is customary for display purposes to work with the magnitude of the transform (a real quantity), which is called the Fourier spectrum or the frequency spectrum:
- i.e.

$$|F(\mu)| = AT\left|\frac{\sin(\pi\mu W)}{(\pi\mu W)}\right|$$

- Figure below shows a plot of |F(μ)| as a function of frequency.



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- The key properties to note are
- The locations of the zeros of both $F(\mu)$ and $|F(\mu)|$ are inversely proportional to the width, W, of the "box" function,
- The height of the lobes decreases as a function of distance from the origin, and
- The function extends to infinity for both positive and negative values of μ.
- These properties are helpful in interpreting the spectra of two-dimensional Fourier transforms of images.

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- Convolution
- This one more building block often used.
- The idea of convolution was learnt already.
- We learned in that section that convolution of two functions involves flipping (rotating by 180°) one function about its origin and sliding it past the other.
- At each displacement in the sliding process, we perform a computation,
- i.e. a sum of products.
- In the present discussion, we are interested in the convolution of two continuous functions, f(t) and h(t), of one continuous variable, t, so we have to use integration instead of a summation.

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- The convolution of these two functions, denoted as before by the operator *, is defined as
- $$f(t) \star h(t) = \int_{-\infty}^{\infty} f(\tau) h(t-\tau)\, d\tau \qquad \ldots(14)$$
- where the – sign indicates the flipping which was done in filtering
- t is the displacement needed to slide one function past the other, and
- τ is a dummy variable that is integrated out.
- We assume for that the functions extend from -∞ to ∞.
- We have seen the basic mechanics of convolution in module 2,
- At the moment, we are interested in finding the Fourier transform of Eq (14)

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- We start with equation

$$\Im\{f(t)\star h(t)\} = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau)\,h(t-\tau)\,d\tau\right]e^{-j2\pi\mu t}\,dt$$

$$= \int_{-\infty}^{\infty} f(\tau)\left[\int_{-\infty}^{\infty} h(t-\tau)\,e^{-j2\pi\mu t}\,dt\right]d\tau$$

- The term inside the brackets is the Fourier transform of h(t - τ).
- We see later that $F\{h(t - \tau)\} = H(\mu)e^{-j2\pi\mu\tau}$, where H(μ) is the Fourier transform of h(t).
- Using this fact in the preceding equation gives us

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$$\Im\{f(t)\star h(t)\} = \int_{-\infty}^{\infty} f(\tau)\left[H(\mu)e^{-j2\pi\mu\tau}\right]d\tau$$

$$= H(\mu)\int_{-\infty}^{\infty} f(\tau)e^{-j2\pi\mu\tau}\,d\tau$$

$$= H(\mu)F(\mu)$$

- We refer to the domain of t as the spatial domain, and the domain of IL as the frequency domain,
- The preceding equation tells us that the Fourier transform of the convolution of two functions in the spatial domain is equal to the product in the frequency domain of the Fourier transforms of the two functions

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- Conversely, if we have the product of the two transforms, we can obtain the convolution in the spatial domain by computing the inverse Fourier transform.
- In other words, f(t) * h(t) and H(u) F(u) are a Fourier transform pair. This result is one-half of the convolution theorem and is written as $$f(t) \star h(t) \Leftrightarrow H(\mu)F(\mu)$$
- The double arrow is used to indicate that the expression on the right is obtained by taking the Fourier transform of the expression on the left, while the expression on the left is obtained by taking the inverse Fourier transform of the expression on the right.
- Following a similar development would result in the other half of the convolution theorem:

$$f(t)h(t) \Leftrightarrow H(\mu) \star F(\mu)$$

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- This states that convolution in the frequency domain is analogous to multiplication in the spatial domain, the two being related by the forward and inverse Fourier transforms, respectively

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- If we have the product of the two transforms, we can obtain the convolution in the spatial domain by computing the inverse Fourier transform

$$f(t) \star h(t) \Leftrightarrow H(\mu)F(\mu)$$

- Convolution in the frequency domain is analogous to multiplication in the spatial domain, the two being related by the forward and inverse Fourier transforms, respectively

$$f(t)h(t) \Leftrightarrow H(\mu) \star F(\mu)$$

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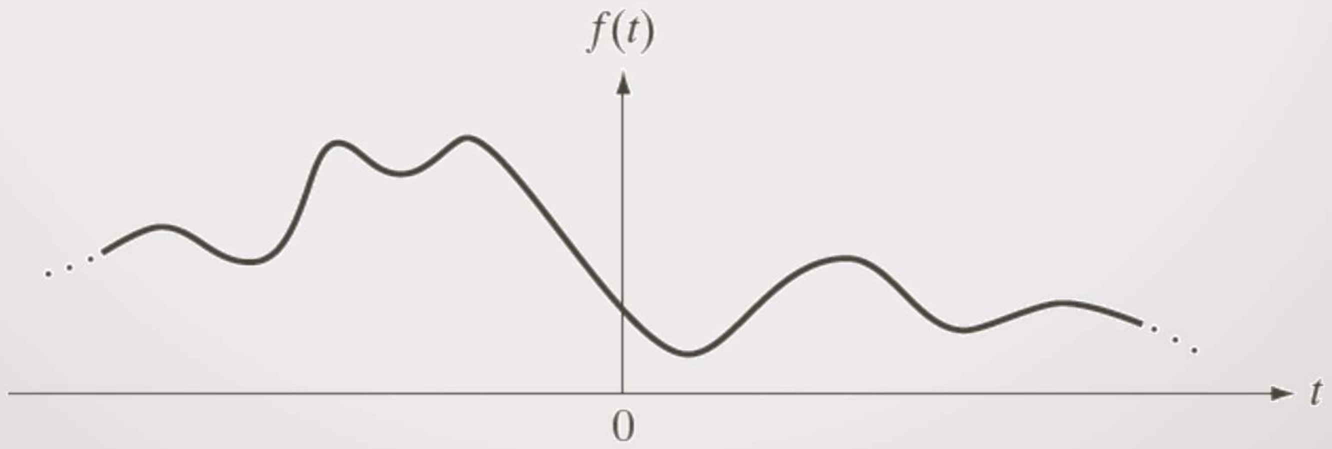
- Sampling and the Fourier Transform of Sampled Functions
- Sampling
- W.k.t. Continuous functions have to be converted into a sequence of discrete values before they can be processed in a computer.
- This is done by using sampling and quantization, as discussed in module 1
- Now we examine sampling in more detail.
- Consider a continuous function, $f(t)$ with $(-\infty < t < \infty)$, that we wish to sample at uniform intervals (ΔT) of the independent variable t.



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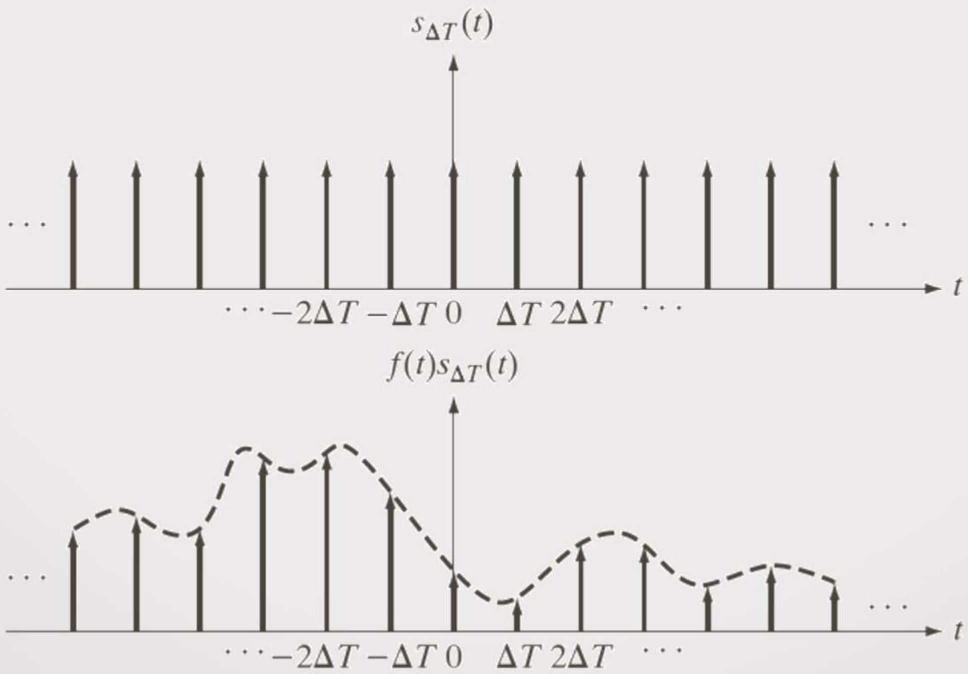
- One simple way of achieving sampling is to multiply f(t) by a sampling function, equal to the train of impulses, separated by ΔT units.
- i.e. $\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)$ ….. (1)

- Each component of this summation is an impulse weighted by the value of f(t) at that location



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- The value of each sample is then given by the "strength" of the weighted impulse, which is obtained by integration.
- That is, the value, f_k, of an arbitrary sample in the sequence is given by

$$f_k = \int_{-\infty}^{\infty} f(t)\,\delta(t - k\Delta T)\,dt$$

- Using the sifting property, we can write

$$f_k = f(k\Delta T)$$

- This equation is valid for all values of k from -∞,..-1,0,1,2,..∞
- Schematically we can show this as…

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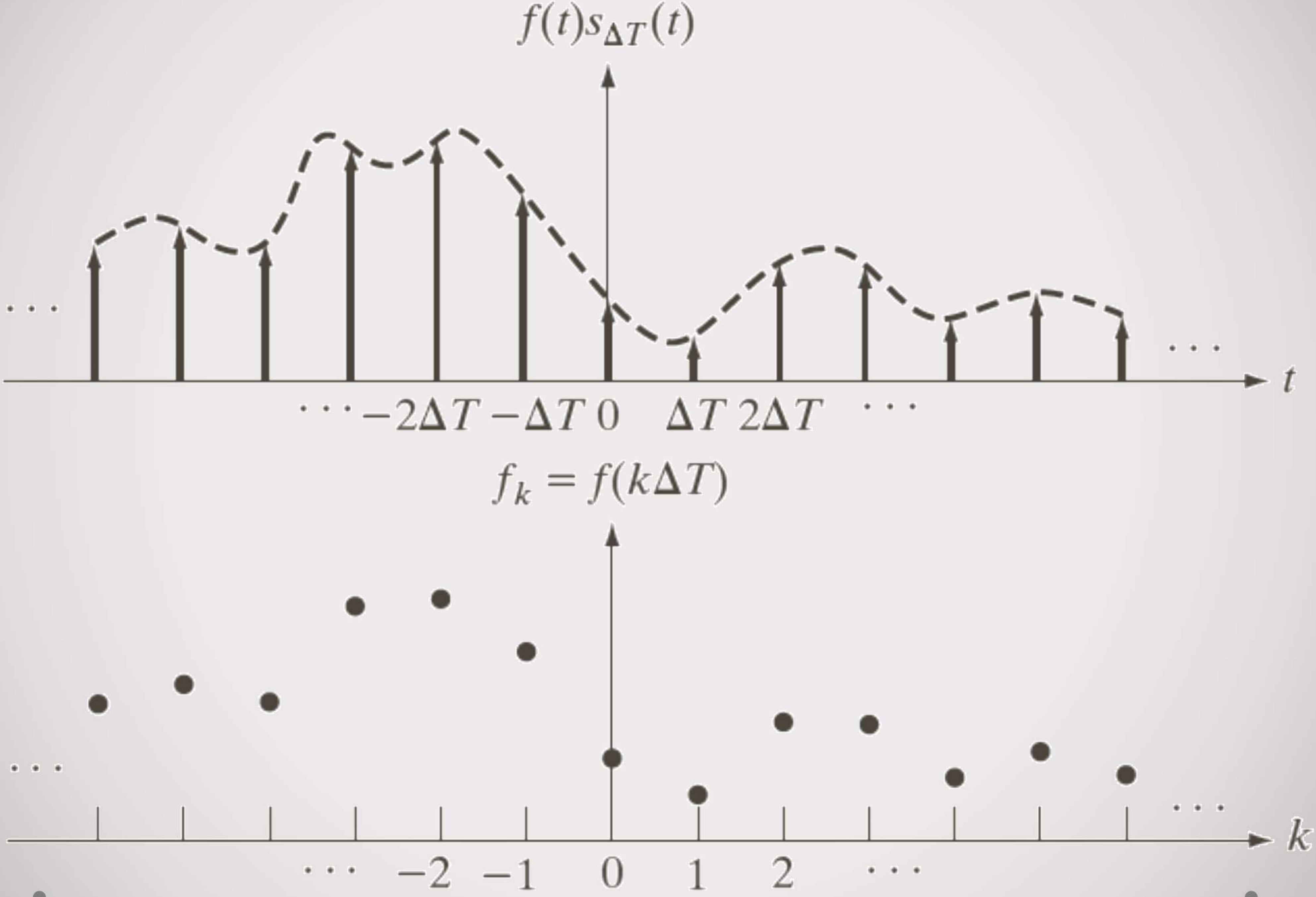

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- The Fourier Transform of Sampled Functions
- Let F(μ) denote the Fourier transform of a continuous function f(t).
- The corresponding sampled function, f (t), is the product of f(t) and an impulse train.
- From the convolution theorem, it is known that, the F.T. of the product of two functions in the spatial domain is the convolution of the transforms of the two functions in the frequency domain. Thus, the Fourier transform, F(μ), of the sampled function f(t) is:

$$\tilde{F}(\mu) = \Im\{\tilde{f}(t)\}$$

$$= \Im\{f(t)s_{\Delta T}(t)\} \qquad \text{.....(2)}$$

$$= F(\mu) \star S(\mu)$$

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- Where S(μ) is the Fourier transform of impulse train and is given by

$$S(\mu) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty}\delta\left(\mu - \frac{n}{\Delta T}\right)$$

- Thus convolution of F(μ) and S(μ) is obtained from

$$\tilde{F}(\mu) = F(\mu) \star S(\mu)$$

$$= \int_{-\infty}^{\infty} F(\tau)S(\mu - \tau)\,d\tau$$

$$= \frac{1}{\Delta T}\int_{-\infty}^{\infty} F(\tau)\sum_{n=-\infty}^{\infty}\delta\left(\mu - \tau - \frac{n}{\Delta T}\right)d\tau$$

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$$\tilde{F}(\mu) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} F(\tau)\,\delta\left(\mu - \tau - \frac{n}{\Delta T}\right) d\tau$$

$$= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} F\left(\mu - \frac{n}{\Delta T}\right)$$

- The last step is obtained by using sifting property
- The summation in the last line of above Eq shows that the Fourier transform F˜(μ) of the sampled function f(t), is an infinite, periodic sequence of copies of F(μ), the transform of the original, continuous function.
- The separation between copies is determined by the value of 1/ΔT.
- Observe that although f˜(t) is a sampled function, its transform F(μ) is continuous because it consists of copies of F(μ) which is a continuous function.

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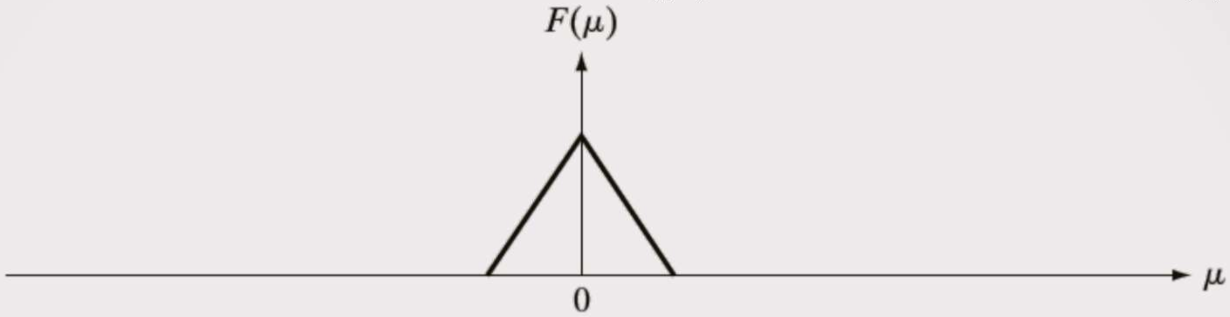
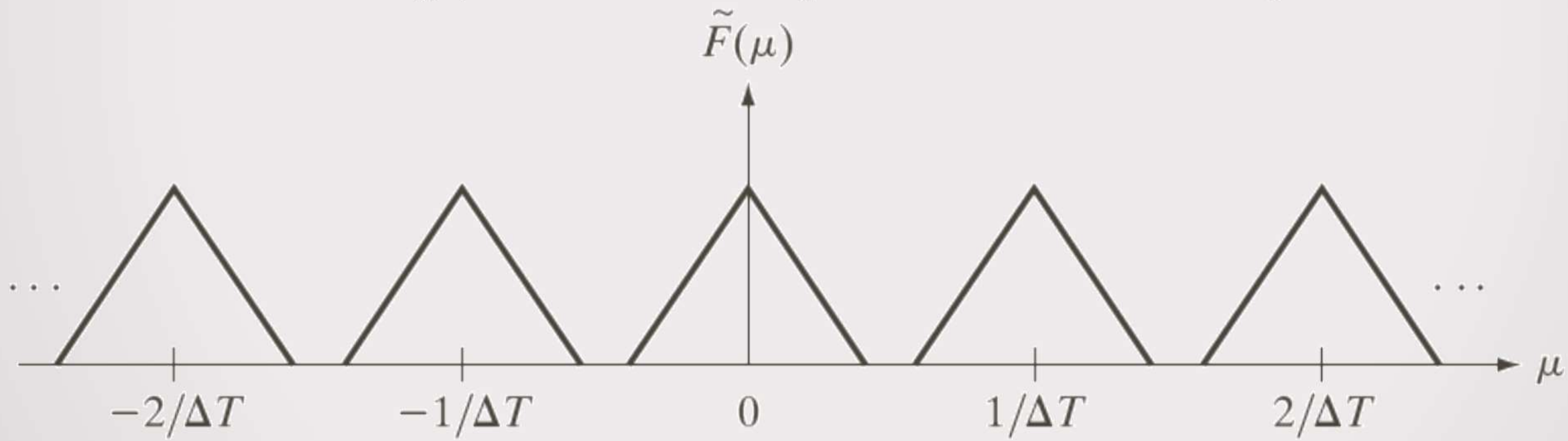
- Previous discussions can be graphically shown as below
- Consider the Fourier Transform F(μ) of a function f(t) plotted as below



- The transform F˜(μ) of the sampled function is given below



- We can see that, enough sampling rate was chosen to provide sufficient separation between periods and thereby preserving F(μ)

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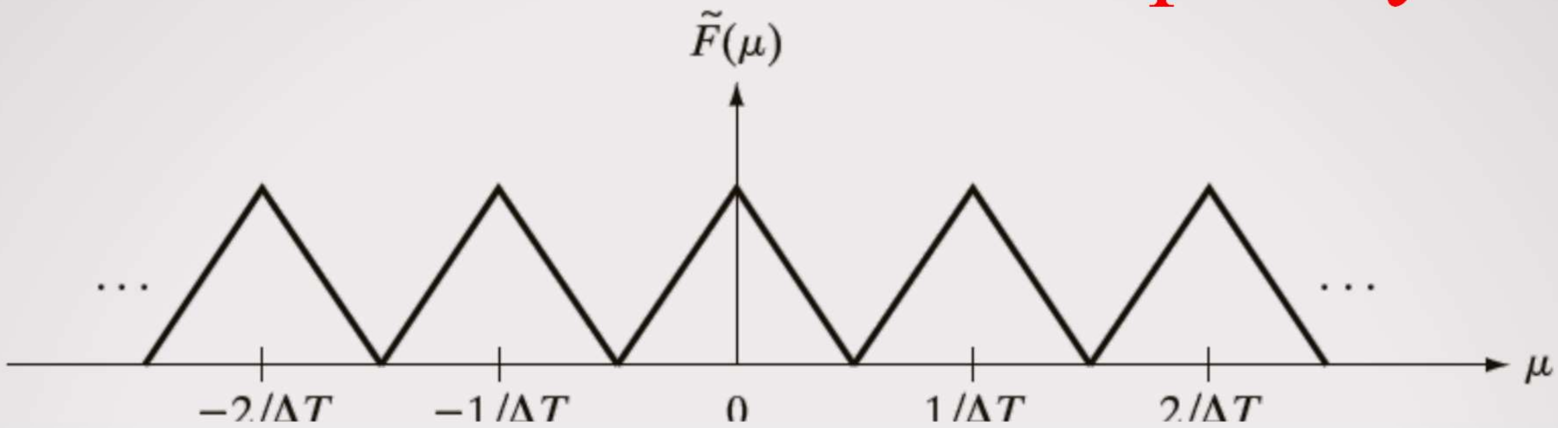
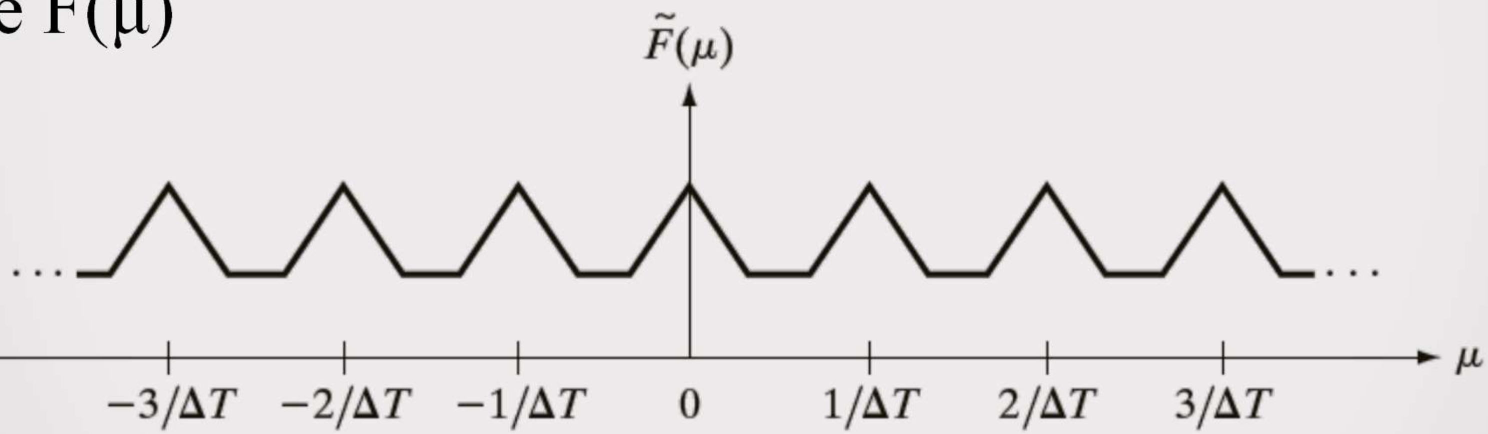


- In this we can see that, sampling rate chosen is just enough to preserve F(μ)



- In this case the sampling rate chosen was below minimum required to maintain distinct copies of F(μ) and thus has failed to preserve F(μ).

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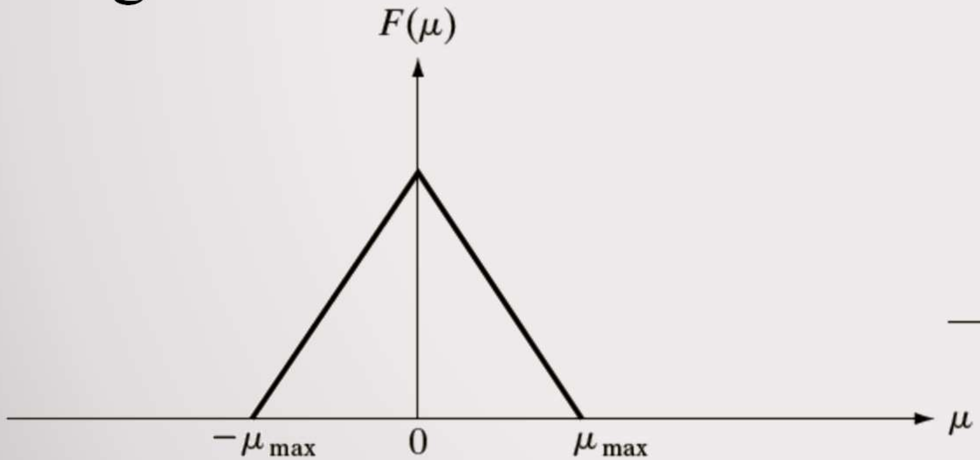
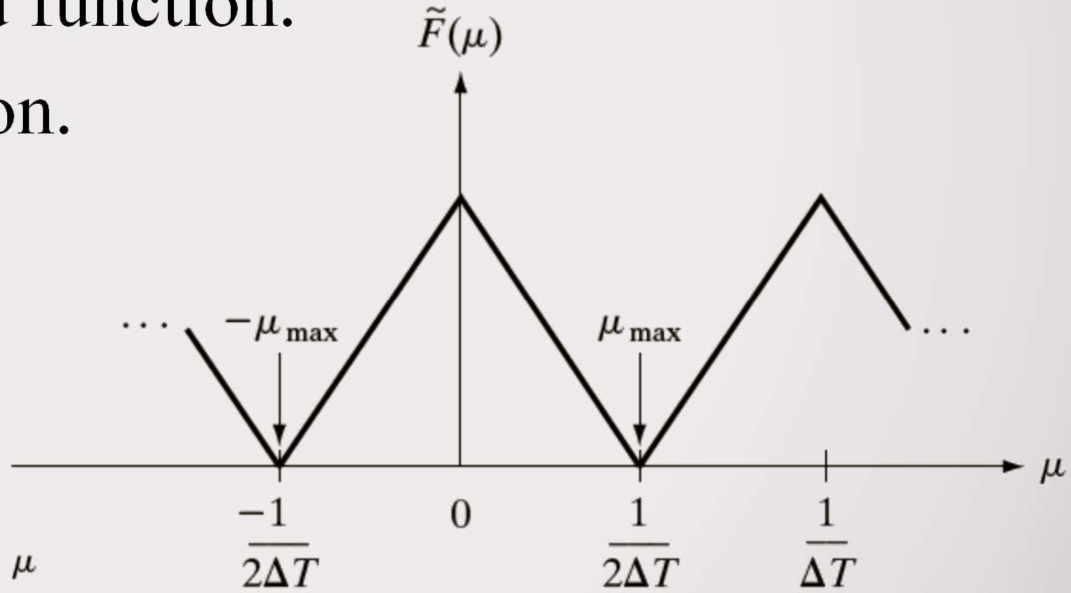
- These three cases are known as over-sampling, critically sampling and under-sampling
- How to choose sampling rate?? - Sampling theorem answers this
- A function f(t) whose Fourier transform is zero for values of frequencies outside a finite interval (band) $[-\mu_{max}, \mu_{max}]$ about the origin is called a band-limited function.
- Figure below is such a function.





- Similarly, Fig. (b) is a more detailed view of the transform of a critically sampled function.

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- A lower value of $1/\Delta T$ would cause the periods in $F(\mu)$ to merge; a higher value would provide a clean separation between the periods.
- We can recover $f(t)$ from its sampled version- if we can isolate a copy of $F(\mu)$ from the periodic sequence of copies of this function contained in $\tilde{F}(\mu)$, the transform of the sampled function $f(t)$.
- We have seen that $F(\mu)$ is a continuous, periodic function with period $1/\Delta T$.
- Therefore, all we need is one complete period to characterize the entire transform.
- This implies that we can recover $f(t)$ from that single period by using the inverse Fourier transform.

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- Extracting a single period that is equal to $F(\mu)$ from $\tilde{F}(\mu)$ is possible if the separation between copies is sufficient
- In figure (b) sufficient separation is guaranteed if $1/2\Delta T > \mu_{max}$ or

$$\frac{1}{\Delta T} > 2\mu_{\max}$$

- This equation indicates that a continuous, band-limited function can be recovered completely from a set of its samples if the samples are acquired at a rate exceeding twice the highest frequency content of the function.
- This result is known as the **sampling theorem**.
- Based on this result we can say that, no information is lost if a continuous, band-limited function is represented by samples acquired at a rate greater than twice the highest frequency content of the function.

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- Conversely, we can say that the maximum frequency that can be "captured" by sampling a signal at a rate $1/\Delta T$ is $\mu_{max} = 1/2\Delta T$.
- Sampling at the Nyquist rate sometimes is sufficient for perfect function recovery, but there are cases in which this leads to difficulties
- Thus, the sampling theorem specifies that sampling must exceed the Nyquist rate.

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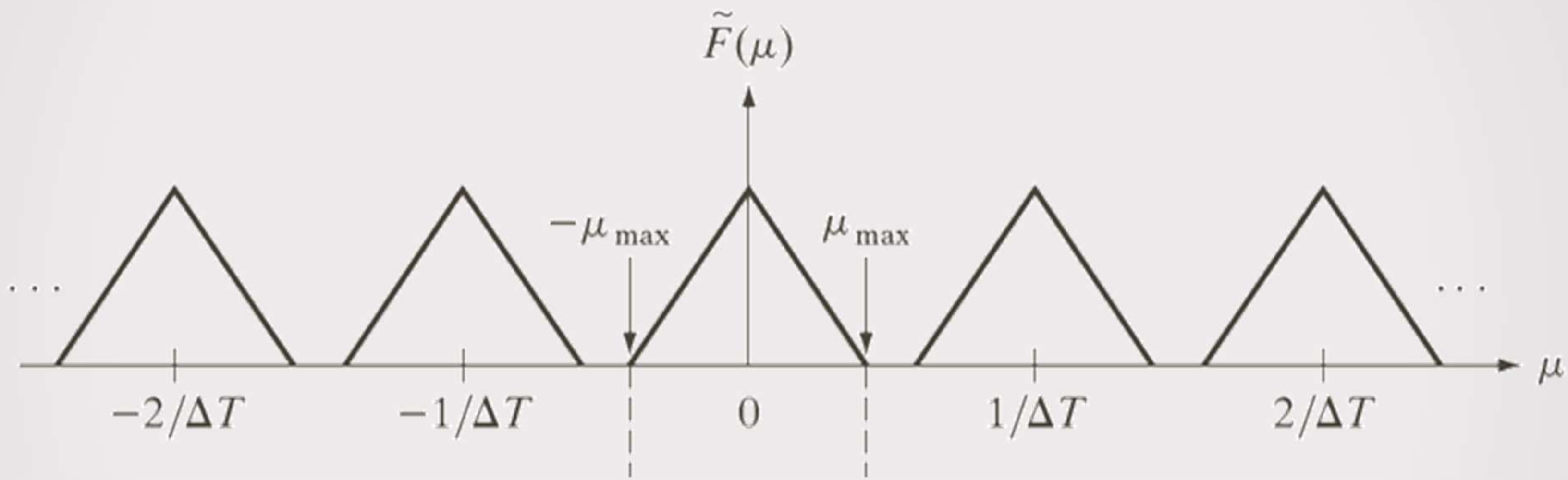
- How to recover $F(\mu)$ from $\tilde{F}(\mu)$
- Consider Fig. below, which shows the Fourier transform of a function sampled at a rate slightly higher than the Nyquist rate.
- 

- Consider another function $H(\mu)$ defined by the equation

$$H(\mu) = \begin{cases} \Delta T & -\mu_{max} \le \mu \le \mu_{max} \\ 0 & \text{otherwise} \end{cases}$$

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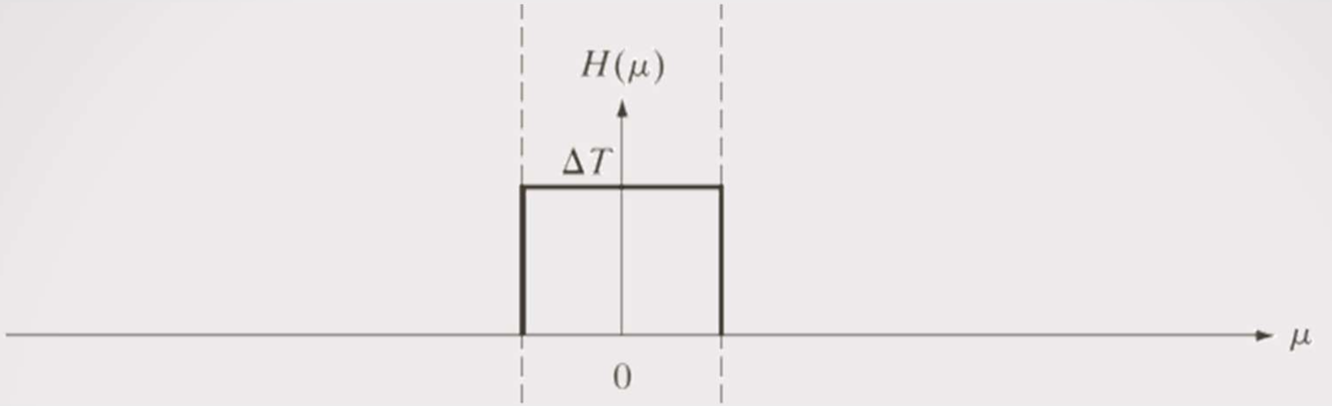
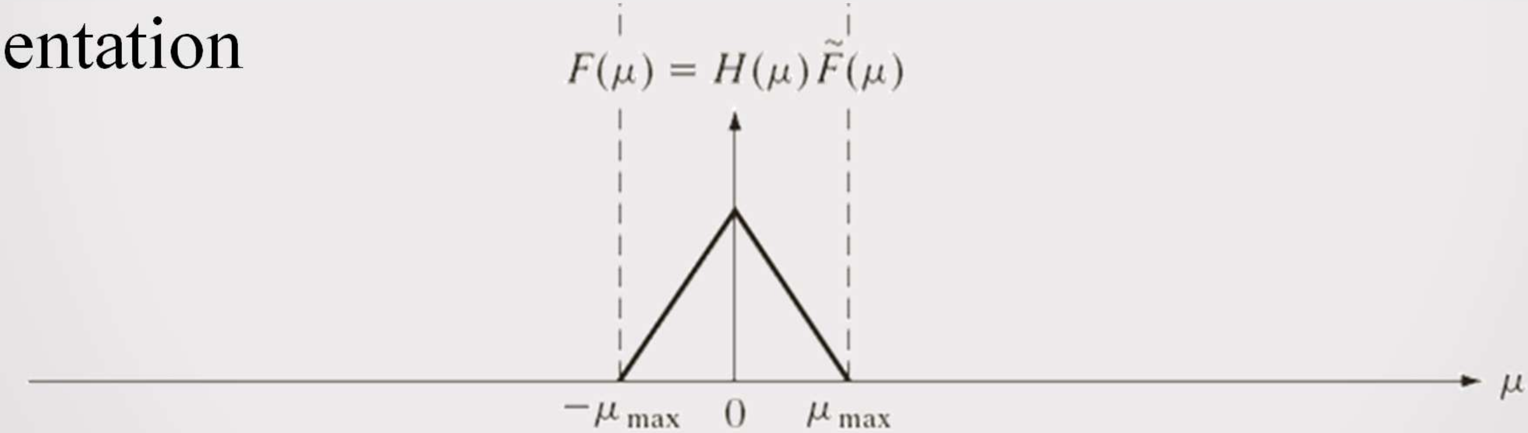
- This can be graphically represented as



- If we multiply these two signals, then we get the following representation



- This is nothing but F(μ) obtained by
-
$$F(\mu) = H(\mu)\tilde{F}(\mu)$$
-

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- Once we have F(μ) we can recover f(t) by using the inverse Fourier transform

$$f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t} d\mu$$

- These equations show that, theoretically it is possible to recover the a band-limited function from samples of the function obtained at a rate exceeding twice the highest frequency content of the function.
- The requirement that f(t) must be band-limited implies that f(t) must extend from 00 to 00, a condition that cannot be met in practice.
- This filter is called low pass filter as it passes lower frequency at the
- low end of the frequency range but it eliminates (filters out) all higher frequencies.
- It is called also an ideal lowpass filter because of its infinitely rapid transitions in amplitude

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- Since they are instrumental in recovering (reconstructing) the original function from its samples, these filters are called reconstruction filters.

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- The Discrete Fourier Transform (OFT) of One Variable
- Here we derive discrete Fourier transform (DFT) starting from basic principles.
- Obtaining the DFT from the Continuous Transform of a Sampled Function
- Fourier transform of a sampled, band-limited function extending from -∞ to ∞ is a continuous, periodic function that also extends from -∞ to ∞.
- In practice, we work with a finite number of samples, and here we derive the DFT corresponding to such finite sample sets
- We have seen an equation which gives the transform, $F^{\sim}(\mu)$ of sampled data in terms of the transform of the original function, but it does not give us an expression for $F^{\sim}(\mu)$ in terms of the sampled function $f^{\sim}(t)$ itself.

Image Enhancement In The Frequency Domain

- We find such equation by using the definition of F.T.
- $$\tilde{F}(\mu) = \int_{-\infty}^{\infty} \tilde{f}(t)\, e^{-j2\pi\mu t}\, dt \qquad (1)$$
- By substituting $\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)$, we get
-

Image Enhancement In The Frequency Domain

$$\begin{aligned}\tilde{F}(\mu) &= \int_{-\infty}^{\infty} \tilde{f}(t) e^{-j2\pi\mu t}\, dt \\ &= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T) e^{-j2\pi\mu t}\, dt \\ &= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\delta(t - n\Delta T) e^{-j2\pi\mu t}\, dt \\ &= \sum_{n=-\infty}^{\infty} f_n e^{-j2\pi\mu n\Delta T} \qquad (2)\end{aligned}$$

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Image Enhancement In The Frequency Domain

- The last step is obtained from the result

$$f_k = \int_{-\infty}^{\infty} f(t)\,\delta(t - k\Delta T)\,dt$$

- We know that, though f_n is a discrete function, its Fourier $F^{\sim}(\mu)$ is continuous and infinitely periodic with period $1/\Delta T$
- Therefore, we need to characterize $F^{\sim}(\mu)$ is for one period, and sampling one period is the basis for the DFT.
- Suppose that we want to obtain M equally spaced samples of $F^{\sim}(\mu)$ taken over the period $\mu = 0$ to $\mu = 1/\Delta T$.
- This is accomplished by taking the samples at the following frequencies:
- $$\mu = \frac{m}{M\Delta T} \qquad m = 0, 1, 2, \ldots, M-1 \qquad (3)$$

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- Substituting this result in equation (2) we get
- $$F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M} \qquad m = 0, 1, 2, \ldots, M-1 \qquad (4)$$
- This is the expression for discrete Fourier transform
- Given a set $\{f_n\}$ consisting of M samples of f(t), Eq. 4 yields a sample set $\{F_m\}$ of M complex discrete values corresponding to the discrete Fourier transform of the input sample set.
- Conversely, given $\{F_m\}$, we can recover the sample set $\{f_n\}$ by using the inverse discrete Fourier transform (IDFT)

$$f_n = \frac{1}{M} \sum_{m=0}^{M-1} F_m e^{j2\pi mn/M} \qquad n = 0, 1, 2, \ldots, M-1$$

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- Eqs. 4 and 5 form a discrete Fourier transform pair which indicates that the forward and inverse Fourier transforms exist for any set of samples whose values are finite.
- Note that neither expression depends explicitly on the sampling interval ΔT nor on the frequency intervals of Eq. 3.
- Therefore, the DFT pair is applicable to any finite set of discrete samples taken uniformly.
- Here we used m and n to denote discrete variables because it is typical to do so for derivations.
- It is more intuitive, especially in two dimensions, to use the notation x and y for image coordinate variables and u and v for frequency variables, where these are understood to be integers.
- Then, Eqs. 4 and 5 become

Image Enhancement In The Frequency Domain

- $$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M} \qquad u = 0, 1, 2, \ldots, M-1 \qquad (6)$$
- and
- $$f(x) = \frac{1}{M} \sum_{u=0}^{M-1} F(u) e^{j2\pi ux/M} \qquad x = 0, 1, 2, \ldots, M-1 \qquad (7)$$
- where we used functional notation instead of subscripts for simplicity.
- Clearly, F(u) == F_m and f(x) = f_n.
- Now onwards we use Eqs. 6 and 7 to denote the 1-D DFT pair.

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- It can be shown that both the forward and inverse discrete transforms are infinitely periodic, with period M.
- That is, $F(\mu) = F(\mu + kM)$ (8)
- and
- $f(x) = f(x + kM)$ (9)
- where k is an integer.
- The discrete form of convolution is given by
- $$f(x) \star h(x) = \sum_{m=0}^{M-1} f(m)h(x-m) \qquad (10)$$
- for x = 0, 1,2, ... , M - 1.

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- Since the functions used in the preceding formulations are periodic, their convolution also is periodic.
- Equation (10) gives one period of the periodic convolution.
- For this reason, this equation often is referred to as circular convolution, and is a direct result of the periodicity of the DFT and its inverse.
- This is in contrast with the convolution we studied earlier, in which values of the displacement, x, were determined by sliding one function completely past the other, and were not fixed to the range [0, M - 1] as in circular convolution.

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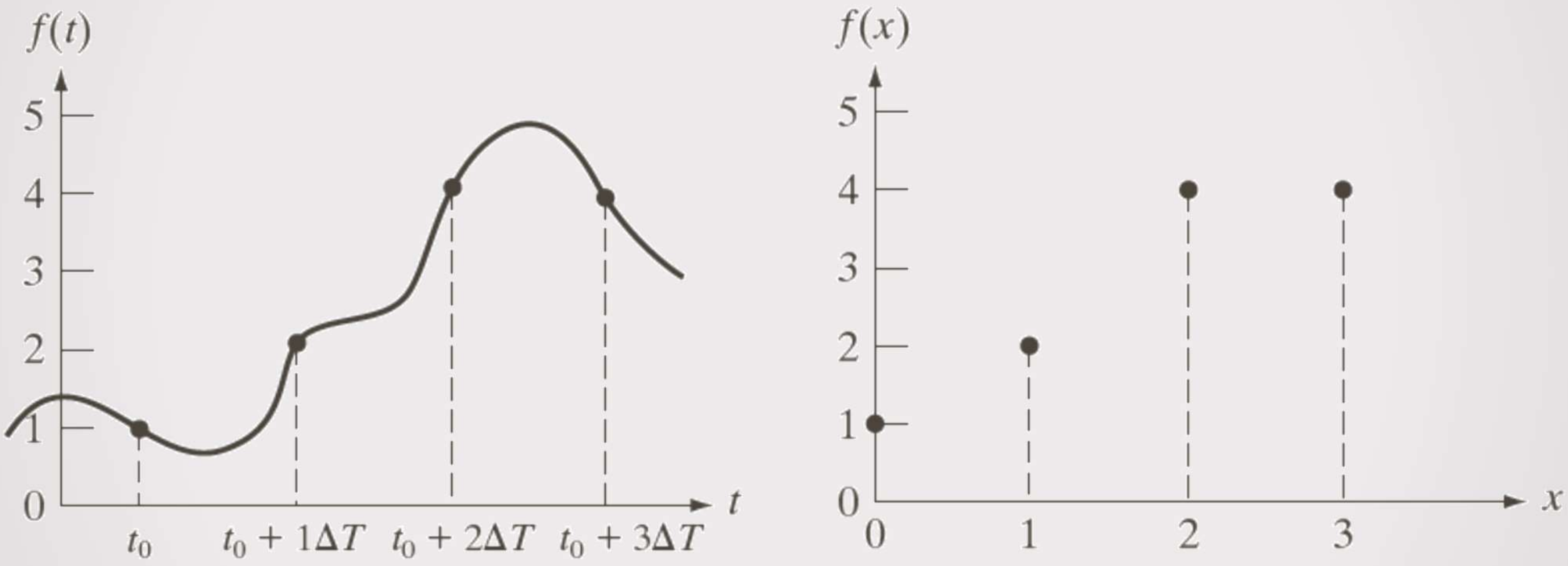
- E.g. : calculation of DFT
- Consider four samples of a continuous function, f(t), taken AT units apart as shown below



- the sampled values in the x-domain are shown in figure b.
- Note that the values of x are 0, 1, 2, and 3, indicating that we could be referring to any four samples of f(t).

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- From equation (6)

$$F(0) = \sum_{x=0}^{3} f(x) = [f(0) + f(1) + f(2) + f(3)]$$

$$= 1 + 2 + 4 + 4 = 11$$

- The next value of F(μ) is F(1) and is obtained by

$$F(1) = \sum_{x=0}^{3} f(x) e^{-j2\pi(1)x/4}$$

$$= 1e^{0} + 2e^{-j\pi/2} + 4e^{-j\pi} + 4e^{-j3\pi/2} = -3 + 2j$$

- Similarly we can obtain Similarly, F(2) = -(1 + 0j) and F(3) = -(3 + 2j).
- Observe that all values of f(x) are used in computing each term of F(μ).

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- Suppose that, we were given F(μ) and were asked to compute its inverse, we would proceed in the same manner, but using the inverse transform

$$f(0) = \frac{1}{4}\sum_{u=0}^{3} F(u)e^{j2\pi u(0)}$$

$$= \frac{1}{4}\sum_{u=0}^{3} F(u)$$

$$= \frac{1}{4}[11 - 3 + 2j - 1 - 3 - 2j]$$

$$= \frac{1}{4}[4] = 1$$

Image Enhancement In The Frequency Domain

- Extension to Functions of Two Variables
- Now we extend concepts learnt earlier to two variables
- The 2-D Impulse and Its Sifting Property
- The impulse, 8(t, z), of two continuous variables, t and z, is defined as
- $$\delta(t, z) = \begin{cases} \infty & \text{if } t = z = 0 \\ 0 & \text{otherwise} \end{cases} \qquad (11\text{-a})$$
- Satisfying the condition that,
- $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(t, z)\, dt\, dz = 1 \qquad (11\text{-b})$$

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- Similar to 1-D, the 2-D impulse also exhibits the sifting property under integration,
- i.e.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)\,\delta(t,z)\,dt\,dz = f(0,0)$$

- more generally for an impulse located at coordinates (t_0, Z_0),
- $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)\,\delta(t-t_0,z-z_0)\,dt\,dz = f(t_0,z_0) \qquad (12)$$
- We can see that the sifting property yields the value of the function f(t, z) at the location of the impulse
- For discrete variables x and y, the 2-D discrete impulse is defined as

$$\delta(x,y) = \begin{cases} 1 & \text{if } x = y = 0 \\ 0 & \text{otherwise} \end{cases}$$

Image Enhancement In The Frequency Domain

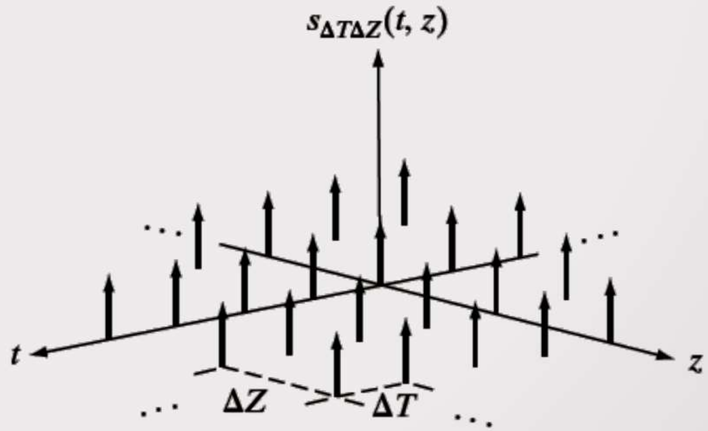
- Sifting property for discrete impulse is defined by
- $$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(x, y)\delta(x, y) = f(0, 0) \qquad (13)$$
- where f(x, y) is a function of discrete variables x and y.
- For an impulse located at coordinates (x_0, y_0) the sifting property is
- $$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(x, y)\delta(x - x_0, y - y_0) = f(x_0, y_0) \qquad (14)$$
- Here also the sifting property of a discrete impulse yields the value of the discrete function f(x, y) at the location of the impulse

Image Enhancement In The Frequency Domain

- The 2-D Continuous Fourier Transform Pair
- Let f(t, z) be a continuous function of two continuous variables, t and z.
- The two-dimensional, continuous Fourier transform pair is given by the expressions
- $$F(\mu, \nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t, z) e^{-j2\pi(\mu t + \nu z)} dt\, dz \qquad (15)$$
- and
- $$f(t, z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\mu, \nu) e^{j2\pi(\mu t + \nu z)} d\mu\, d\nu \qquad (16)$$
- where μ and v are the frequency variables.
- When referring to images, t and z are interpreted to be continuous spatial variables.
- The variables μ and v belong to the continuous frequency domain

mage Enhancement In The Frequency Domain

- Two-Dimensional Sampling and the 2-D Sampling Theorem
- Sampling in two dimensions can be modeled using the sampling function (2-D impulse train):
- $$s_{\Delta T \Delta Z}(t, z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(t - m\Delta T, z - n\Delta Z) \qquad (17)$$
- where ΔT and ΔZ are the separations between samples along the t- and z-axis of the continuous function f(t, z).
- Equation (17) describes a set of periodic impulses extending infinitely along the two axes



- As in the 1-D case, multiplying f(t, z) by $s_{\Delta T \Delta Z}(t, z)$ yields the sampled function

Image Enhancement In The Frequency Domain

- Function f(t, z) is said to be band-limited if its Fourier transform is outside a rectangle established by the intervals $[-\mu_{max}, \mu_{max}]$ and $[-v_{max}, v_{max}]$:
- that is, $$F(\mu, \nu) = 0 \quad \text{for } |\mu| \geq \mu_{max} \text{ and } |\nu| \geq \nu_{max} \tag{18}$$
- 2-D Sampling theorem
- The two-dimensional sampling theorem states that a continuous, band-limited function f(t, z) can be recovered with no error from a set of its samples if the sampling intervals are
- $$\Delta T < \frac{1}{2\mu_{max}} \quad \text{and} \quad \Delta Z < \frac{1}{2\nu_{max}}$$
- i.e. no information is lost if a 2-D, band-limited, continuous function is represented by samples acquired at rates greater than twice the highest frequency content of the function in both the wand v-directions.

Image Enhancement In The Frequency Domain

- The 2-D Discrete Fourier Transform and Its Inverse
- 2-D discrete Fourier transform (DFT) is given by the equation

$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M + vy/N)}$$

- where f(x, y) is a digital image of size M x N.
- Above Eq. must be evaluated for values of the discrete variables u and v in the ranges u = 0, 1, 2, ... , M - 1 and v = 0, 1, 2, . , .. N – 1
- Given the transform F(u, v), we can obtain f(x, y) by using the inverse discrete Fourier transform (IDFT):

$$f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi(ux/M + vy/N)}$$

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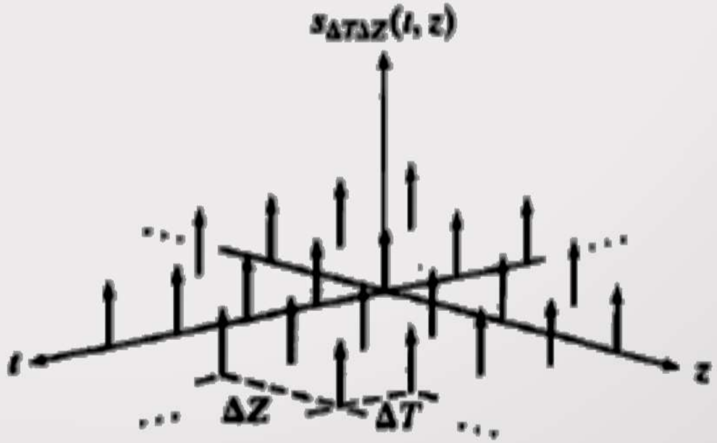
- for x = 0,1,2, ... , M - 1 and y = 0,1,2, ... , N -1.
- Above two equations form the 2-D discrete Fourier transform pair.

- Some Properties of the 2-D Discrete Fourier Transform
- Relationships Between Spatial and Frequency Intervals
- Suppose that a continuous function f(t, z) is sampled to form a digital image, f(x, y), consisting of M x N samples taken in the t and z-directions, respectively.
- Let ΔT and ΔZ denote the separations between samples as shown below



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- Then, the separations between the corresponding discrete, frequency domain variables are given by
- $$\Delta u = \frac{1}{M\Delta T} \quad \text{and} \quad \Delta v = \frac{1}{N\Delta Z}$$
- Note that the separations between samples in the frequency domain are inversely proportional both to the spacing between spatial samples and the number of samples
- Translation and Rotation
- It can be shown by direct substitution into the equations of 2-D DFT and 2-D IDFT that the Fourier transform pair satisfies the following translation properties

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$$f(x, y)e^{j2\pi(u_0x/M + v_0y/N)} \Leftrightarrow F(u - u_0, v - v_0)$$

$$f(x - x_0, y - y_0) \Leftrightarrow F(u, v)e^{-j2\pi(x_0u/M + y_0v/N)}$$

- That is, multiplying f(x, y) by the exponential shown shifts the origin of the DFT to (u_0, v_0) and,
- Also multiplying F(u, v) by the negative of that exponential shifts the origin of f(x, y) to (x_0, y_0).
- Note that translation has no effect on the magnitude (spectrum) of F(u, v).

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- Using the polar coordinates
- $x = r\cos\theta$, $y = r\sin\theta$, $u = w\cos\varphi$, $v = w\sin\varphi$ results in the following transform pair:

$$f(r, \theta + \theta_0) \Leftrightarrow F(\omega, \varphi + \theta_0)$$

- This indicates that rotating f(x, y) by an angle θ_0 rotates F(u, v) by the same angle.
- Also rotating F(u, v) rotates f(x, y) by the same angle
- Periodicity
- As in the 1-D case, the 2-D Fourier transform and its inverse are infinitely periodic in the u and v directions; that is,

Image Enhancement In The Frequency Domain

$$F(u, v) = F(u + k_1M, v) = F(u, v + k_2N) = F(u + k_1M, v + k_2N)$$

and

$$f(x, y) = f(x + k_1M, y) = f(x, y + k_2N) = f(x + k_1M, y + k_2N)$$

- where k1 and k2 are integers
- The periodicities of the transform and its inverse are important issues in the implementation of DFT-based algorithms.
- Consider the 1-D spectrum as shown in Fig. (a).

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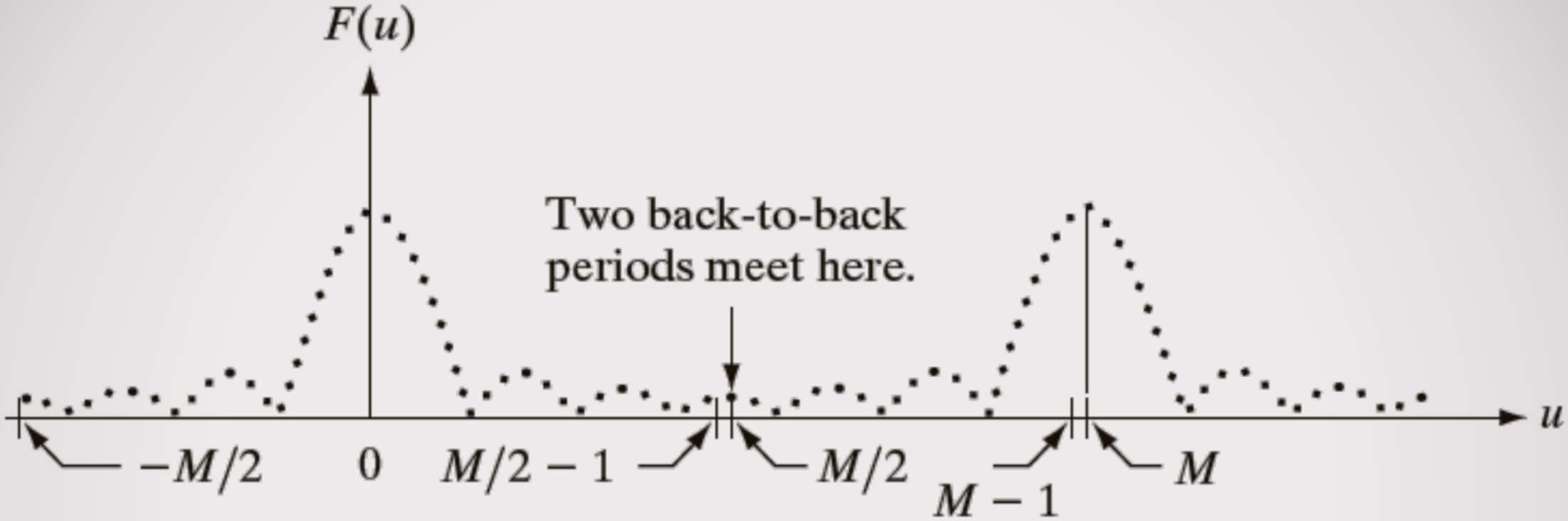

- The transform data in the interval from 0 to M -1 consists of two back-to-back half periods meeting at point M/2.
- For display and filtering purposes, it is more convenient to have in this interval a complete period of the transform in which the data are contiguous, as in Fig. (b)

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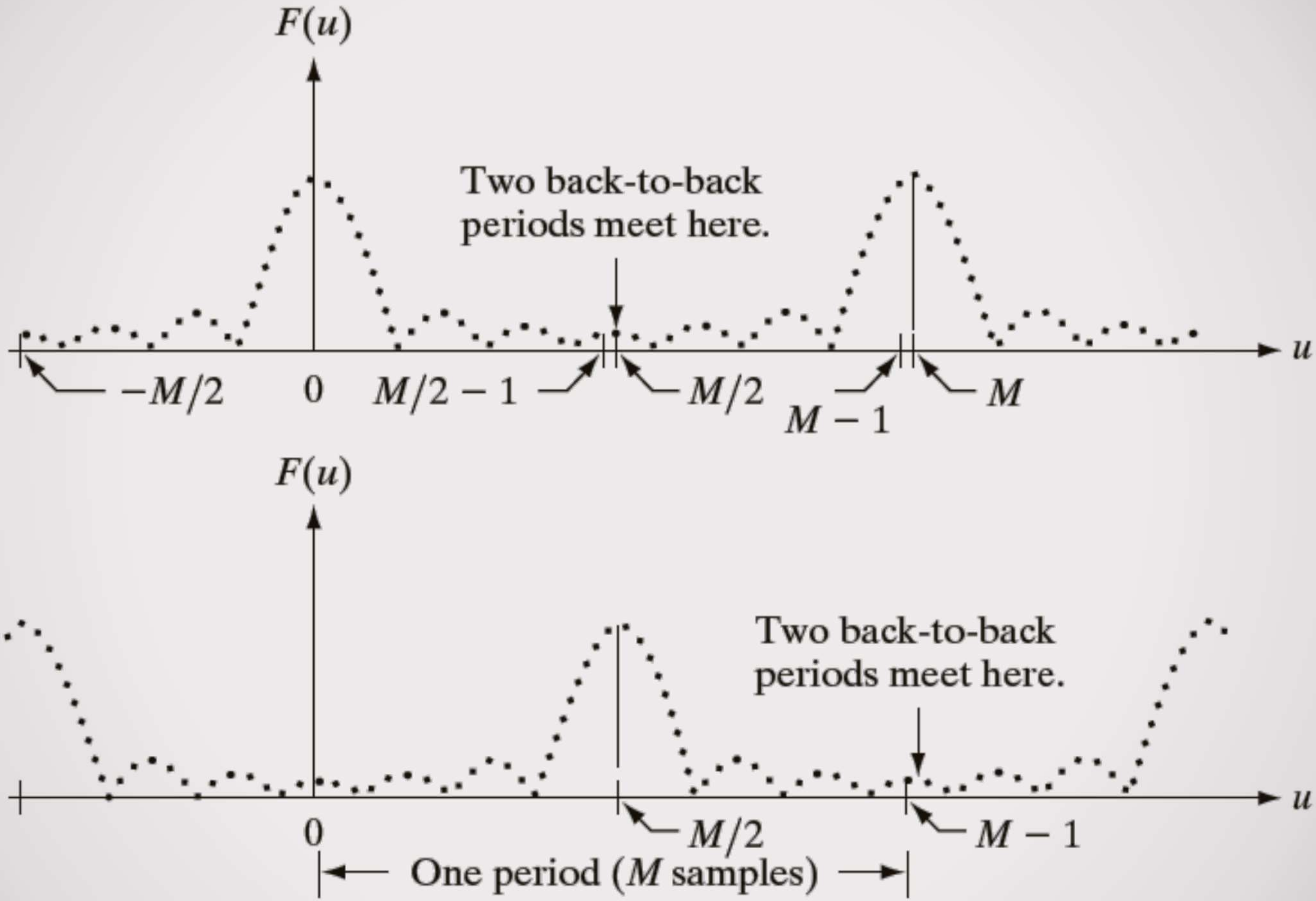


Image Enhancement In The Frequency Domain

- From translation property we can write that

$$f(x)e^{j2\pi(u_0 x/M)} \Leftrightarrow F(u - u_0)$$

- In other words, multiplying f(x) by the exponential term shown shifts the data so that the origin, F(0), is located at u_0.
- If we let $u_0 = M/2$, the exponential term becomes $e^{j\pi x}$ which is equal to $(-1)^x$ because x is an integer
- Thus we get

$$f(x)(-1)^x \Leftrightarrow F(u - M/2)$$

- Thus multiplying f(x) by $(-1)^x$ shifts the data so that F(0) is at the center of the interval [0, M - 1], which corresponds to Fig. (b), as desired

Image Enhancement In The Frequency Domain

- Basics of filtering in frequency domain
- The equations for discrete DFT and IDFT are as below

$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M + vy/N)} \qquad (4.5\text{-}15)$$

$$f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi(ux/M + vy/N)} \qquad (4.5\text{-}16)$$

- There is some relationship between frequency components of F.T> and spatial features of the image
- Since frequency is directly related to the spatial rate of change, we can relate the frequencies in the F.T. with the patterns of the intensity variations in the image.

Image Enhancement In The Frequency Domain

- When we express DFT in polar form we get

$$F(u, v) = |F(u, v)|e^{j\phi(u,v)} \quad (4.6\text{-}15)$$

where the magnitude

$$|F(u, v)| = \left[R^2(u, v) + I^2(u, v)\right]^{1/2} \quad (4.6\text{-}16)$$

is called the *Fourier* (or *frequency*) *spectrum*, and

$$\phi(u, v) = \arctan\left[\frac{I(u, v)}{R(u, v)}\right] \quad (4.6\text{-}17)$$

- In the transform we have access to magnitude and phase angle.
- Visual analysis of phase angle is not very useful
- Magnitude or spectrum provides some useful guidelines as to gross characteristics of the image from which spectrum was generated

Image Enhancement In The Frequency Domain

- Filtering in frequency domain is based on modifying the F.T. to achieve a specific objective and then computing the inverse DFT to get back the image
- For a given digital image f(x, y) of size M X N, basic filtering equation will be of the form

$$g(x, y) = \Im^{-1}[H(u, v)F(u, v)]$$

- where $\Im^{-1}$ is the IDFT
- F(u, v) is DFT of the image
- H(u, v) is the filtering function
- g(x, y) is the output image
- Functions F, H and g are arrays of size M X N, same as f(x, y)

Image Enhancement In The Frequency Domain

- The product F(u, v)H(u, v) is obtained using array multiplication
- Specification of H(u, v) is simplified by using functions that are symmetric about their center.
- This required that, F(u, v) also to be centered
- This is done by multiplying the input image by $(-1)^{x+y}$ before computing the transform
- One of the simplest filter is with H(u, v) with 0 at the center of the transform and 1 elsewhere.
- This filter will reject the dc term in the transform and pass all the other terms of F(u, v)

Image Enhancement In The Frequency Domain

- There is a known result about averaging..

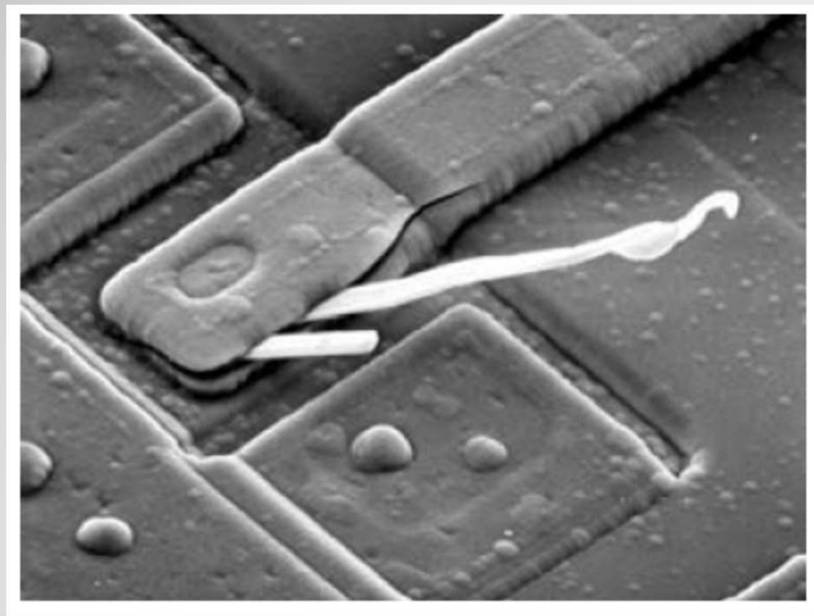
$$F(0,0) = MN\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)$$

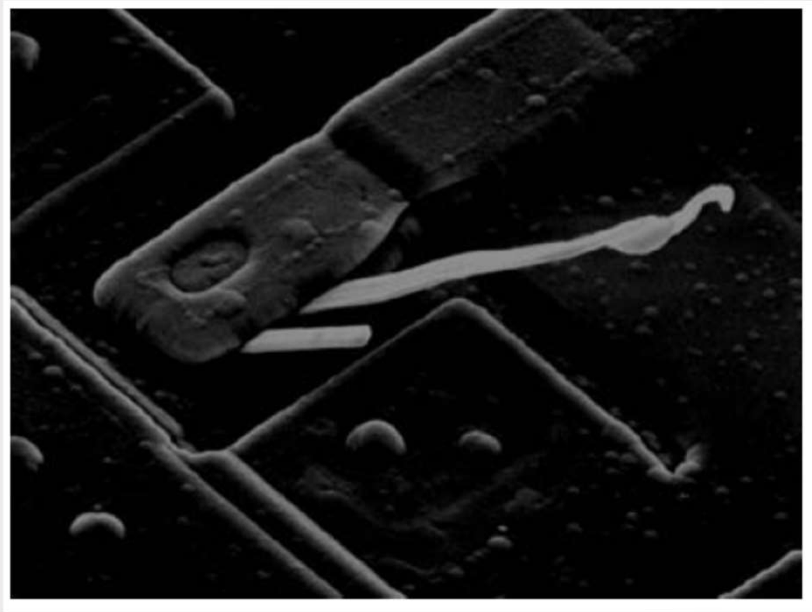
$$= MN\bar{f}(x,y) \qquad (4.6\text{-}21)$$

where $\bar{f}$ denotes the average value of f. Then,

- From the above equation, it is clear that, average intensity of the image is the DC component
- So setting this value to zero will reduce the average intensity of the output image to zero.
- This can be seen in the following figure

Image Enhancement In The Frequency Domain





- The output is darker than the original image
- Average of zero implies the presence of negative intensities.
- In the output for viewing purpose, all the negative intensities are clipped to value 0

Image Enhancement In The Frequency Domain

- Typically low frequencies in the transform are related to slow varying intensity component of the image
- E.g.- walls of the room, cloudless sky etc
- High frequencies are related to sharp transitions in the intensities
- E.g. edges, noise etc.
- Thus we expect that, a filter H(u, v) which attenuates the high frequencies and allows low frequencies (LPF) would blur the image
- Also an HPF would enhance the sharp details of the image but there will be reduction in the contrast of the image

Image Enhancement In The Frequency Domain

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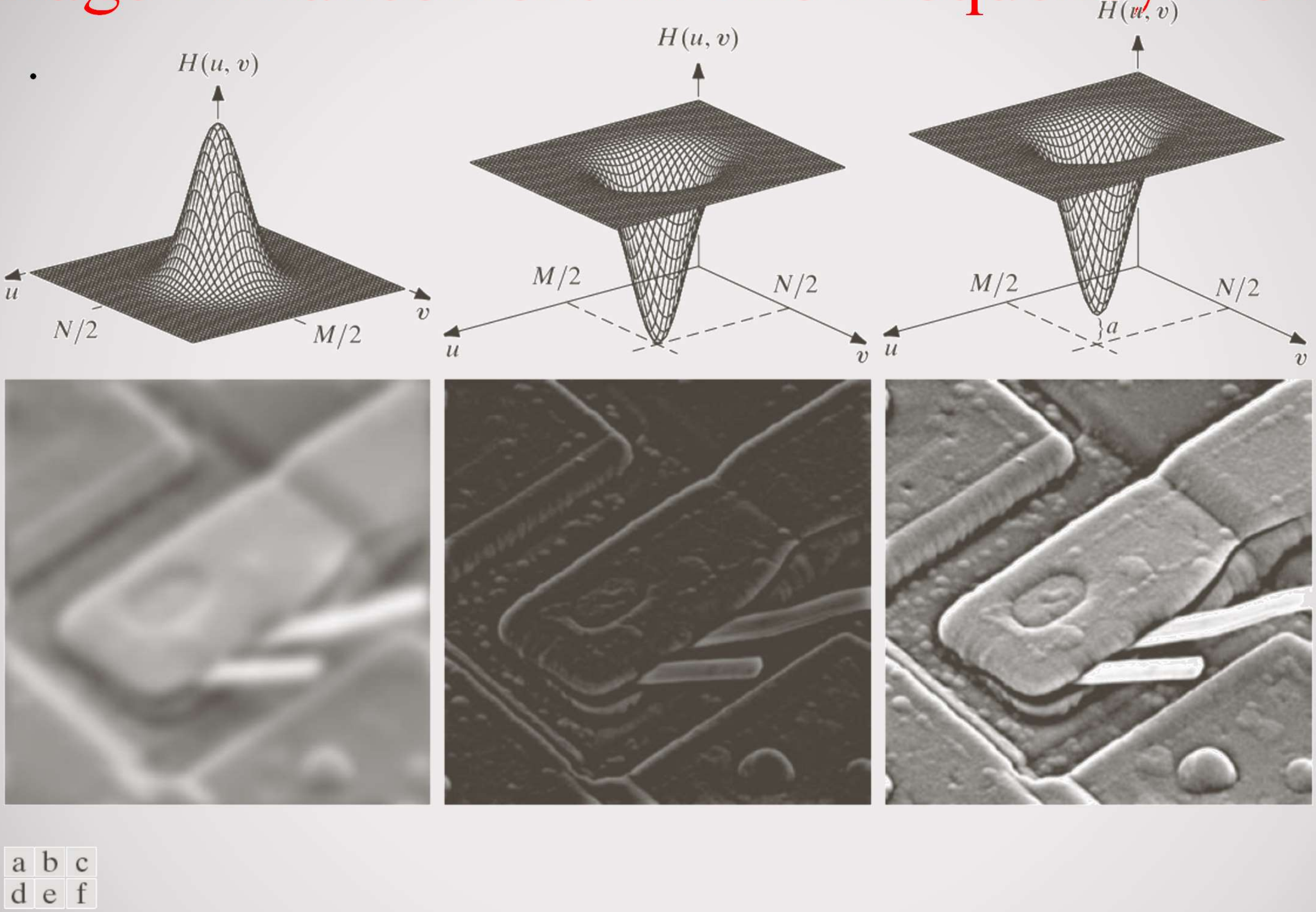


FIGURE 4.31 Top row: frequency domain filters. Bottom row: corresponding filtered images obtained using Eq. (4.7-1). We used $a = 0.85$ in (c) to obtain (f) (the height of the filter itself is 1). Compare (f) with Fig. 4.29(a).

Image Enhancement In The Frequency Domain

- In the last filter, a constant a is added
- It does not affect the sharpening of the image but it prevents the elimination of DC term and thereby preserving the tonality of the image
- By convolution theorem, we know that, multiplication in frequency domain is same as convolution in spatial domain.
- If the functions under consideration are not properly padded this leads to an error called wraparound error
- Consider an image

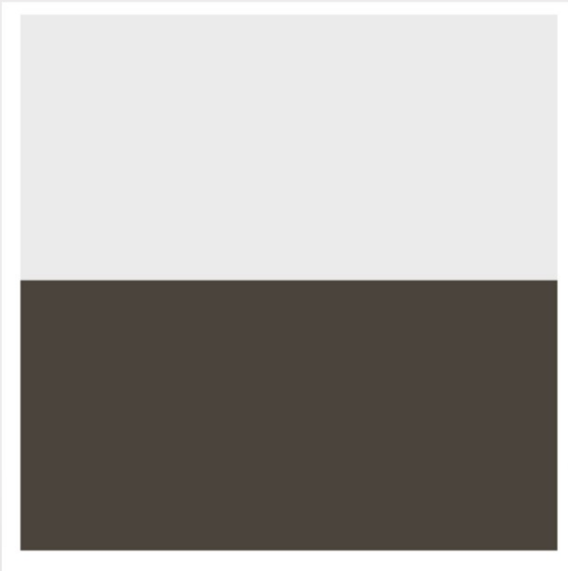
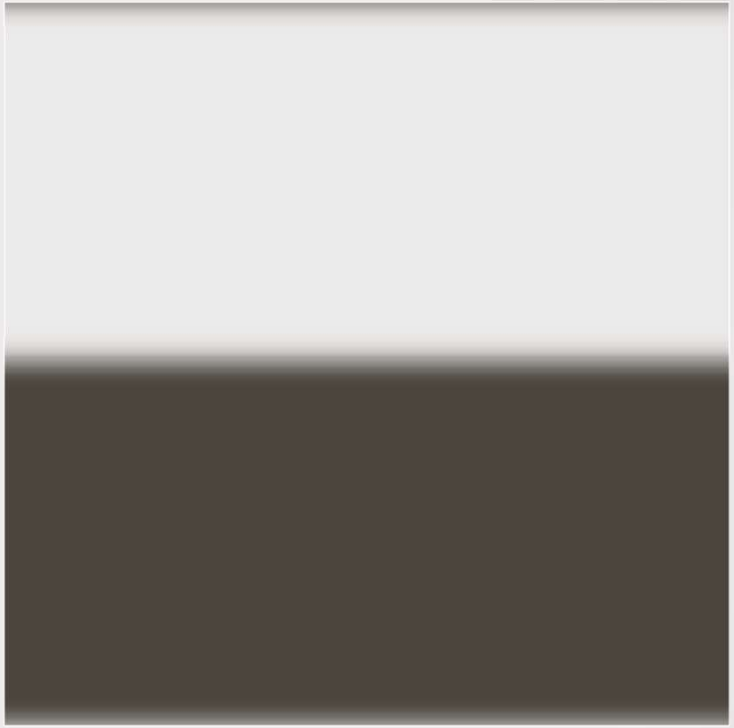


Image Enhancement In The Frequency Domain

- When this is applied to a Gaussian LPF without padding we get the following image



- Is image blurred??
- But blurring is not uniform
- Vertical edges are not blurred.
- If we apply padding suitably then??.

Image Enhancement In The Frequency Domain

- How much padding is needed??

- The resulting padded images are of size $P \times Q$. If both arrays are of the same size, $M \times N$, then we require that

$$P \geq 2M - 1 \tag{4.6-31}$$

and

$$Q \geq 2N - 1 \tag{4.6-32}$$

Image Enhancement In The Frequency Domain

- Note that, DFT algorithms tend to execute fast with arrays of even size
- So choose P and Q as the smallest even integers that satisfy above conditions

- How to do padding if the filter is given in frequency domain??

- Though there are a few issues we use padding to the size of P X Q
- Summary ..

Image Enhancement In The Frequency Domain

1. Given an input image $f(x, y)$ of size $M \times N$, obtain the padding parameters P and Q from Eqs. (4.6-31) and (4.6-32). Typically, we select $P = 2M$ and $Q = 2N$.
2. Form a padded image, $f_p(x, y)$, of size $P \times Q$ by appending the necessary number of zeros to $f(x, y)$.
3. Multiply $f_p(x, y)$ by $(-1)^{x+y}$ to center its transform.
4. Compute the DFT, $F(u, v)$, of the image from step 3.
5. Generate a real, symmetric filter function, $H(u, v)$, of size $P \times Q$ with center at coordinates $(P/2, Q/2)$.† Form the product $G(u, v) = H(u, v)F(u, v)$ using array multiplication; that is, $G(i, k) = H(i, k)F(i, k)$.

Image Enhancement In The Frequency Domain

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6. **Obtain the processed image:**

$$g_p(x, y) = \left\{\text{real}\left[\Im^{-1}[G(u, v)]\right]\right\}(-1)^{x+y}$$

where the real part is selected in order to ignore parasitic complex components resulting from computational inaccuracies, and the subscript p indicates that we are dealing with padded arrays.

7. Obtain the final processed result, $g(x, y)$, by extracting the $M \times N$ region from the top, left quadrant of $g_p(x, y)$.

Image Enhancement In The Frequency Domain

- Correspondence between filtering in spatial and frequency domains.
- W.k.t. convolution theorem connects filtering in spatial and frequency domains
- We have seen that, in frequency domain, filtering is done by multiplying filter function H(u, v) with F(u, v) the F.T. of the image
- Suppose that, a filter H(u,v) is given to us and we need to find its spatial domain equivalent
- If we let f(x, y) = δ(x, y) this gives us F(u, v) = 1
- Then the filtered output will be $\Im^{-1}\{H(u, v)\}$.
- This is nothing but the inverse transform of the frequency domain filter which results in the corresponding filter in spatial domain

Image Enhancement In The Frequency Domain

- Also we can say, given a spatial filter we obtain its frequency domain representation by taking forward F.T. of the filter
- Thus we can say these two filters form a Fourier Transform pair

$$h(x, y) \Leftrightarrow H(u, v)$$

- As the filter h(x, y) can be obtained from the response of frequency domain filter to an impulse, this is called as impulse response of H(u, v)
- As the discrete implementation of the above equation are finite, these are also called as Finite Impulse response(FIR) filters.
- While discussing spatial domain convolution we have seen that, convolution could be done on functions of different size

Image Enhancement In The Frequency Domain

- But when we speak about convolution in the context of DFT it needs functions to be of same size
- In practice, we prefer to implement convolution filtering using spatial domain convolution equation due to speed and ease of implementation
- But filtering concepts are more intuitive in frequency domain
- One way to take advantages of both domains is to specify a filter in frequency domain, compute its IDFT and then use the resulting spatial filter as guide for constructing smaller spatial masks
- Now let us use Gaussian filter to illustrate this
- Filters based on Gaussian functions have a special property that, both forward and inverse transforms of Gaussian functions are Real Gaussian functions

Image Enhancement In The Frequency Domain

- Let H(u) denote 1-D Gaussian filter in frequency domain

$$H(u) = Ae^{-u^2/2\sigma^2} \qquad (4.7\text{-}5)$$

- Where σ is the std. devialtion
- Corresponding filter in the spatial domain is obtained by taking inverse FT of H(u, v)
- Thus we get $h(x) = \sqrt{2\pi}\sigma A e^{-2\pi^2\sigma^2 x^2}$ (4.7-6)
- These two equations have some important features
- They are Fourier transform pairs. Both components of which are Gaussian and real – so no need to worry about complex numbers
- Secondly the functions behave reciprocally. i.e. when H(u) has broader profile (large value of σ) h(x) has a narrow profile

Image Enhancement In The Frequency Domain

- As H(u) approaches infinity, h(x) tends towards impulse
- The plots of Gaussian low pass filter in frequency domain and corresponding low pass filter in the spatial domain are shown below .

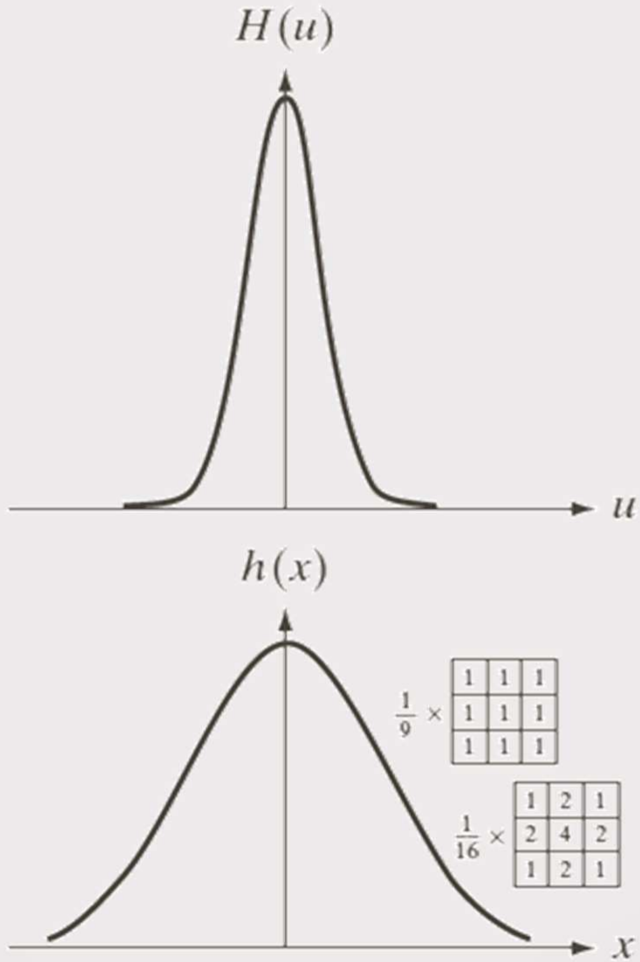

Image Enhancement In The Frequency Domain

- Image smoothing using frequency domain filters
- We know that, edges or any other sharp intensity transitions such as noise in an image contribute significantly to the high frequency content of its Fourier transform
- Thus in frequency domain, smoothing(blurring) is achieved by suppressing the high frequency components
- This is called as low pass filtering
- Here we focus on three types of filters
  - Ideal (very sharp filtering)
  - Gaussian (Very smooth filtering)
  - Butterworth (has an entity called filter order)–
  - If it is higher – this filter approaches Ideal filter
  - For lower values it approaches Gaussian filter

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- All filtering here follows the procedure shown below

1. Given an input image $f(x, y)$ of size $M \times N$, obtain the padding parameters P and Q from Eqs. (4.6-31) and (4.6-32). Typically, we select $P = 2M$ and $Q = 2N$.
2. Form a padded image, $f_p(x, y)$, of size $P \times Q$ by appending the necessary number of zeros to $f(x, y)$.
3. Multiply $f_p(x, y)$ by $(-1)^{x+y}$ to center its transform.
4. Compute the DFT, $F(u, v)$, of the image from step 3.
5. Generate a real, symmetric filter function, $H(u, v)$, of size $P \times Q$ with center at coordinates $(P/2, Q/2)$.† Form the product $G(u, v) = H(u, v)F(u, v)$ using array multiplication; that is, $G(i, k) = H(i, k)F(i, k)$.

Image Enhancement In The Frequency Domain

6. Obtain the processed image:

$$g_p(x, y) = \left\{\text{real}\left[\Im^{-1}[G(u, v)]\right]\right\}(-1)^{x+y}$$

where the real part is selected in order to ignore parasitic complex components resulting from computational inaccuracies, and the subscript p indicates that we are dealing with padded arrays.

7. Obtain the final processed result, $g(x, y)$, by extracting the $M \times N$ region from the top, left quadrant of $g_p(x, y)$.

- H(u,v) are discrete functions of size P x Q – meaning that, frequency variables are in the range u= 1, 2 , .. P-1 and v = =1, 2, .. Q-1
- .

Image Enhancement In The Frequency Domain

- Ideal Low Pass filters:
- A 2-D low pass filter, that passes without attenuation all frequencies within a circle of radius D0 from the origin and cuts off all frequencies outside this circle is called an ideal low pass filter
- It is specified by the function

$$H(u, v) = \begin{cases} 1 & \text{if } D(u, v) \leq D_0 \\ 0 & \text{if } D(u, v) > D_0 \end{cases} \qquad (4.8\text{-}1)$$

- .where D0 is a positive constant
- D(u, v) is the distance between a point (u, v) in the frequency domain and the center of the frequency rectangle and is given by

$$D(u, v) = \left[(u - P/2)^2 + (v - Q/2)^2\right]^{1/2} \qquad (4.8\text{-}2)$$

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- P and Q are padded sizes as seen earlier
- Figure below shows the perspective plot of H(u, v) along with filter displayed as image

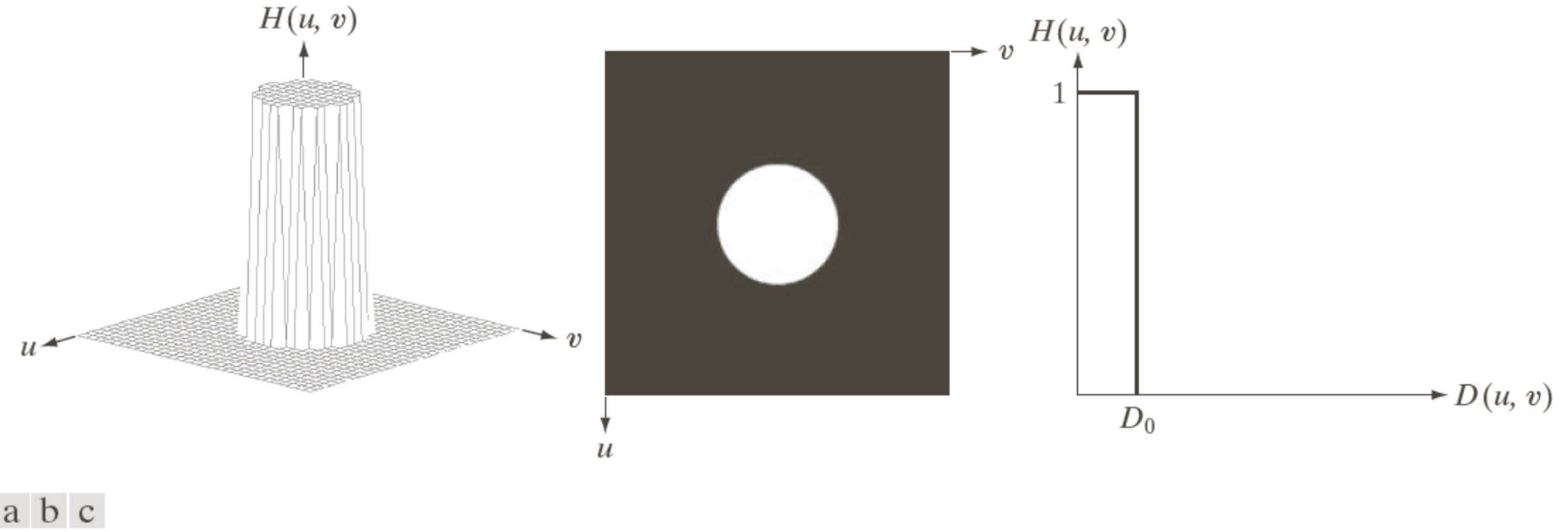

FIGURE 4.40 (a) Perspective plot of an ideal lowpass-filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.

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- The name ideal means, all frequencies on or inside the circle of radius D_0 are passed with no attenuation and all the other frequencies outside the circle are completely attenuated (filtered out)
- The point of transition from H(u, v) = 1 to 0 is called the cut off frequency
- In the figure cut off frequency is D_0
- This type of sharp cut off frequencies are not possible to implement in electronic components but can be simulated in software
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Image Enhancement In The Frequency Domain

- The LPF introduced here are compared by studying their behavior as a function of same cut off frequencies
- One way to establish a set of standard loci is to compute circles that enclose specified amounts of Total image power P_T
- This is obtained by adding the components of the power spectrum of padded image at each point (u, v) for u = 0, 1, 2 …P-1 and v =0, 1, 2, .. Q-1
- i.e.

$$P_T = \sum_{u=0}^{P-1} \sum_{v=0}^{Q-1} P(u, v) \qquad (4.8\text{-}3)$$

- Where P(u, v) is power spectrum and is given by

Image Enhancement In The Frequency Domain

the *power spectrum* is defined as

$$P(u, v) = |F(u, v)|^2 = R^2(u, v) + I^2(u, v) \qquad (4.6\text{-}18)$$

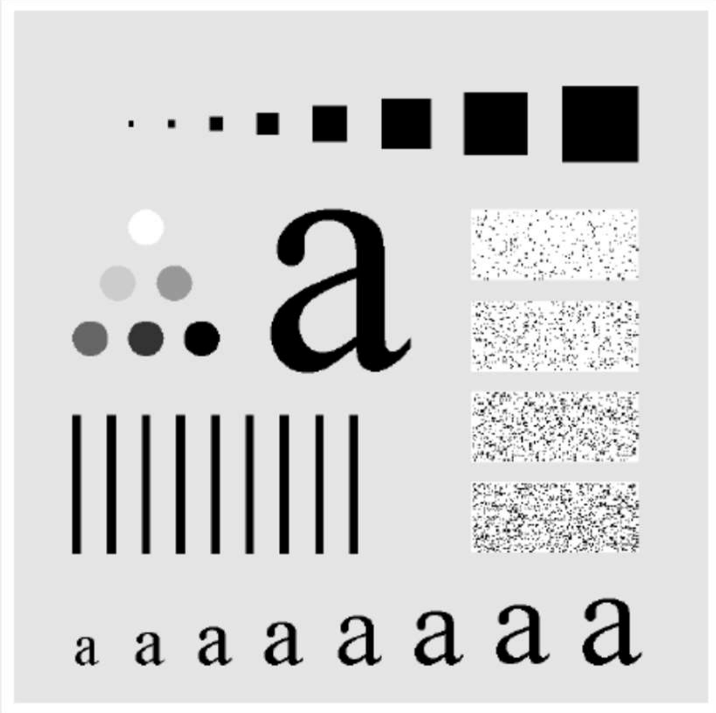
R and I are the real and imaginary parts of $F(u, v)$ and all computations are carried out for the discrete variables $u = 0, 1, 2, \ldots, M - 1$ and $v = 0, 1, 2, \ldots, N - 1$.

- If the DFT is centered a circle, with radius D_0 with origin at the center of the frequency rectangle encloses α percent of the power, where

$$\alpha = 100\left[\sum_u \sum_v P(u, v)/P_T\right] \qquad (4.8\text{-}4)$$

Image Enhancement In The Frequency Domain

- Note that summation is taken over values of (u, v) that lie inside the circle and also on the boundary
- Consider a test pattern image as shown below



- Various D0 values taken are 10, 30, 60, 160 and 460 pixels

Imag… …Domain

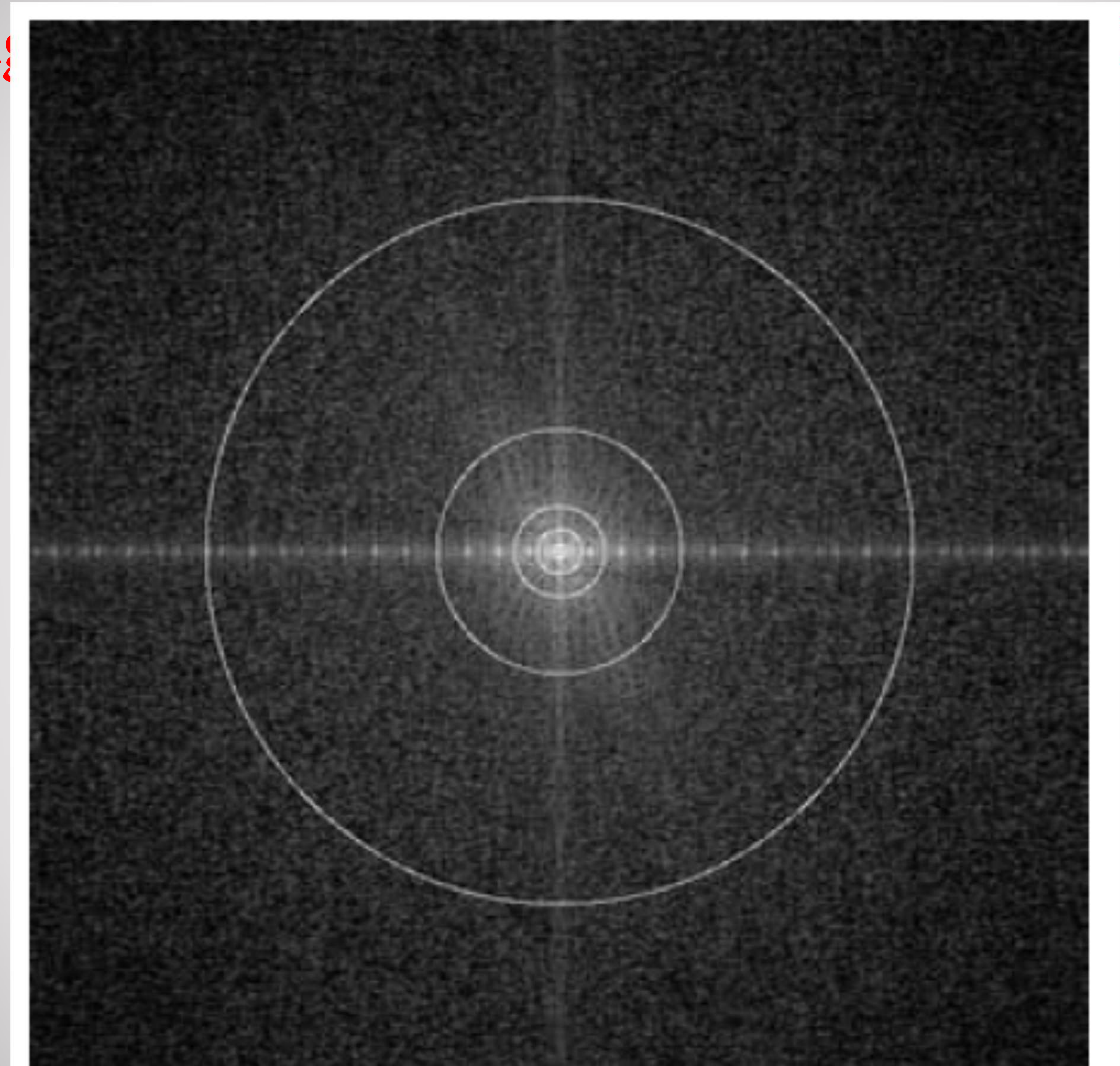
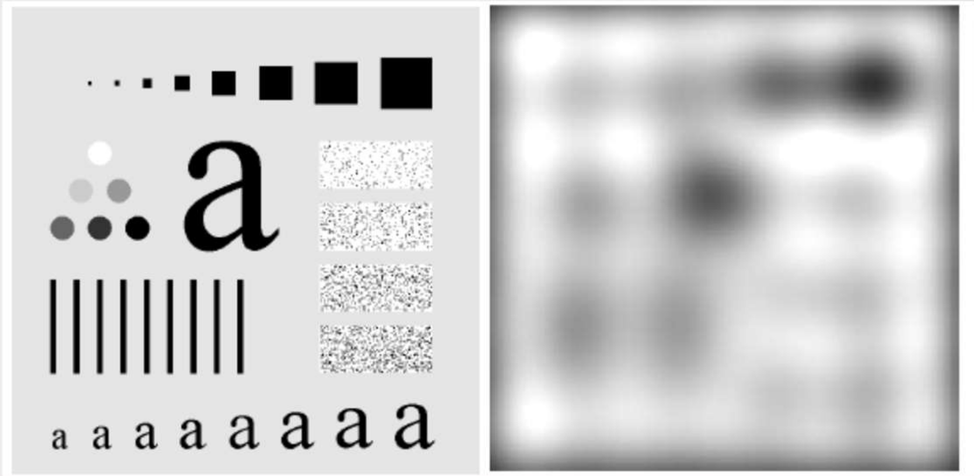


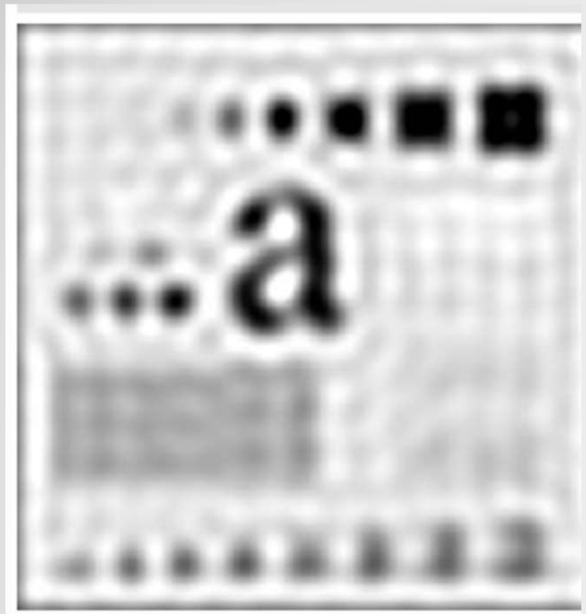
Image Enhancement In The Frequency Domain

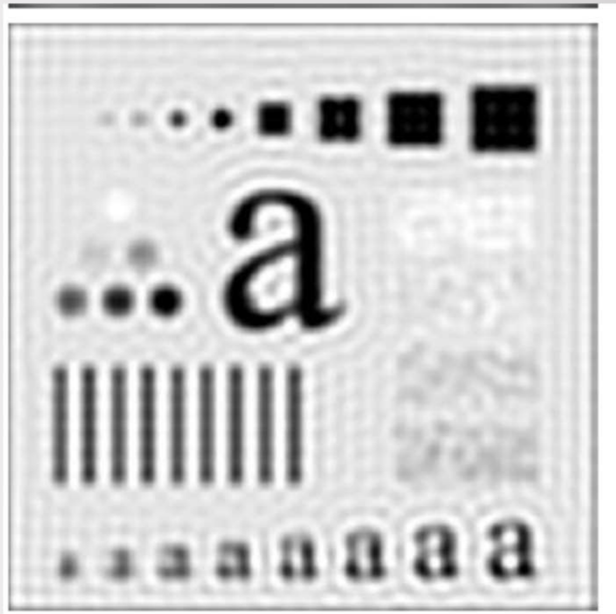
- These circles enclose α percent of image power for α =87.0, 93.1, 95.7, 97.8 and 99.2 % respectively
- Now let us apply this ILPF to the image of test pattern with the above mentioned radii

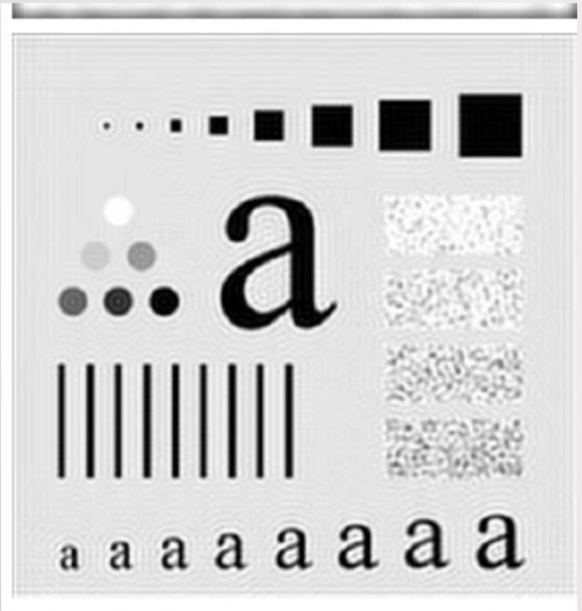


- This output is obtained with radius of 10
- This is useless for all practical purposes. Severe blurring here indicates that, most of the sharp detail information is contained in the 13% of the power removed by the filter
- As radius increases, less power is filtered our resulting in less blur

Image Enhancement In The Frequency Domain







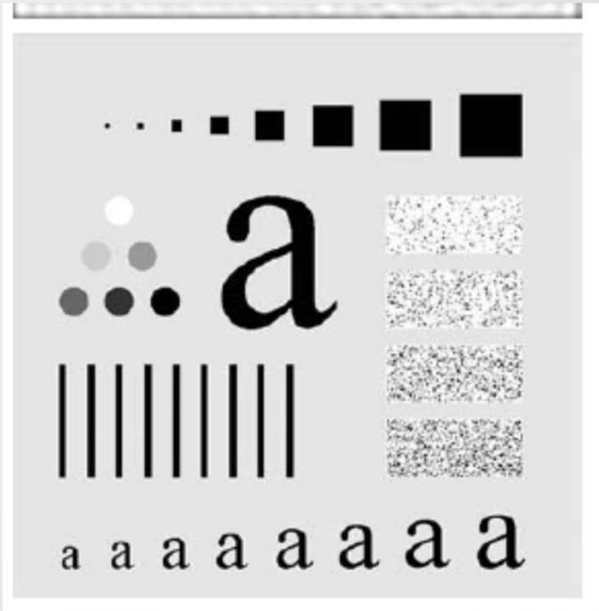


Image Enhancement In The Frequency Domain

- We can see that image becomes more and more finer as amount of high frequency component removed decreases
- The ILPF is not practical. But their study will be useful for development of filter concepts

- Butterworth Lowpass Filter
- The transfer function of Butterworth LPF of the order n with cut off frequency D_0 is given by

$$H(u, v) = \frac{1}{1 + [D(u, v)/D_0]^{2n}} \qquad (4.8\text{-}5)$$

Image Enhancement In The Frequency Domain

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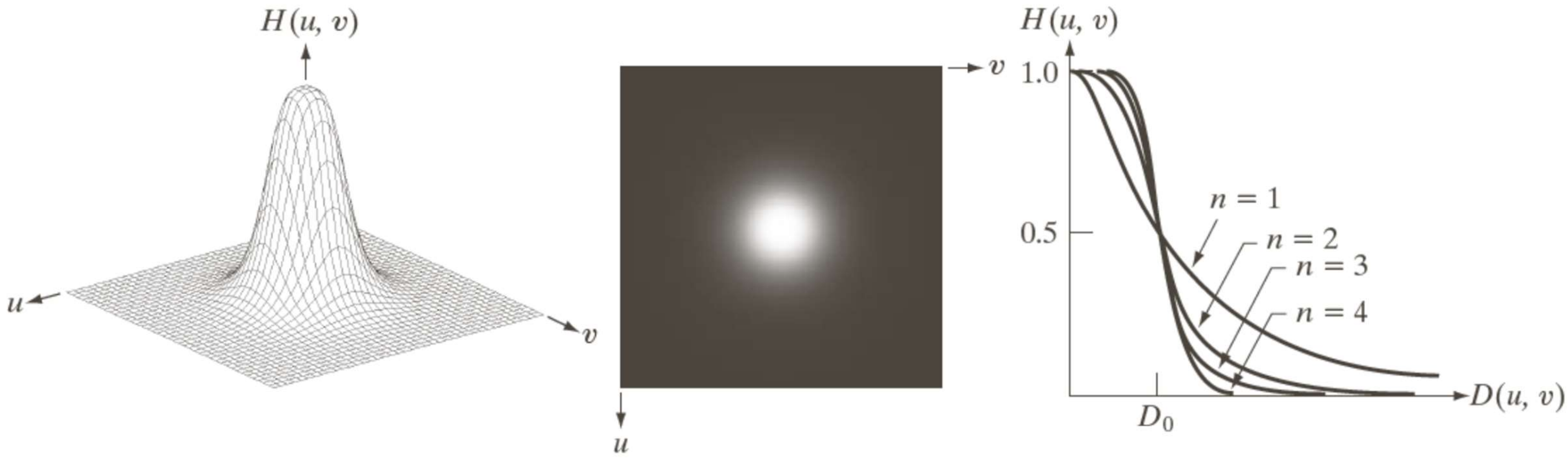

FIGURE 4.44 (a) Perspective plot of a Butterworth lowpass-filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.